

Non periodic Ishibashi states : the $su(2)$ and $su(3)$ affine theories

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Abstract

We consider the $su(2)$ and $su(3)$ affine theories on a cylinder, from the point of view of their discrete internal symmetries. To this end, we adapt the usual treatment of boundary conditions leading to the Cardy equation to take the symmetry group into account. In this context, the role of the Ishibashi states from all (non periodic) bulk sectors is emphasized. This formalism is then applied to the $su(2)$ and $su(3)$ models, for which we determine the action of the symmetry group on the boundary conditions, and we compute the twisted partition functions. Most if not all data relevant to the symmetry properties of a specific model are hidden in the graphs associated with its partition function, and their subgraphs. A synoptic table is provided that summarizes the many connections between the graphs and the symmetry data that are to be expected in general.

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1 Introduction

It is by now well-known that the content of two-dimensional conformal field theories (CFT) is best exposed on surfaces with distinct topology, an idea that goes back to Cardy [1, 2]. In the same way open and closed strings are related to each other, the study of a CFT on a surface with boundaries (a cylinder say) is deeply connected with its formulation on a torus. Bulk data and boundary data are inextricably related and subjected to numerous cross consistency constraints [4, 5, 6]. Aspects of these connections can be most elegantly expressed in terms of graphs [7, 8]: on a cylinder, their nodes label the consistent boundary conditions and their adjacency matrices yield the partition functions, whereas on the torus, their spectra code the diagonal part of modular invariants.

Partition functions are fundamental objects characterizing a specific CFT. On the torus, they are modular invariant and yield the field content in the periodic sector. On the cylinder, they similarly give the field content in the presence of boundary conditions. Torus and cylinder partition functions are not independent: the Cardy equation [3] relates the boundary states and the boundary partition functions to the periodic Hilbert space in the bulk.

The presence of a discrete symmetry within a CFT can be probed by allowing non trivial monodromies of the fields along non contractible loops [9], leading to twisted partition functions. On the torus, these partition functions must transform covariantly under the modular group, thereby relating the bulk Hilbert spaces of the various twisted sectors. On the cylinder, the analogous statement is a generalized Cardy equation, which relates the boundary Hilbert spaces, and the action of the symmetry group on them, with the bulk Hilbert spaces in the twisted sectors.

For the affine $\widehat{su}(2)$ and $\widehat{su}(3)$ theories, the (discrete) symmetry group of each model has been determined in [10] from an analysis on the torus. The object of this article is to extend the study to the cylinder, following a method presented in [11] for the Virasoro minimal models. As on the torus, where most of the information about the symmetry group and its action on the fields was encoded in the graphs, we find here that the same relationship holds on the cylinder.

Therefore, instead of giving the detailed, case by case results of our analysis, it is more appealing to say how they can be extracted from basic data of the graph. To this end we provide in Table 2 a chart summarizing our observations on the way the symmetry data can be read off from the graph data. This table is a central result of this paper.

Before going to the cylinder, we start in Section 2 by briefly recalling the highlights of the analysis on the torus. A number of observations, made in the Virasoro minimal models and in the $su(2)$ and $su(3)$ affine models and which turn out to be crucial for the rest of the analysis, are explained (and included in the synoptic table).

Section 3 is devoted to a general discussion of the boundary conditions and of the cylinder partition functions when the theory has a symmetry group. The main result here is that a boundary condition invariant under a symmetry subgroup can be expanded on the Ishibashi states taken from the torus sectors twisted by that subgroup. It directly leads to a sequence

of Cardy equations.

Relying in an essential way on the observations recalled in Section 2, the fourth section and an appendix make explicit the action of the symmetry group on the boundary conditions for the affine models considered here.

For the same models, Section 5 solves the Cardy equations for the boundary conditions which have a non trivial isotropy group, thereby determining the way this group acts on the corresponding boundary Hilbert spaces. We show that the relevant information can be extracted from subgraphs, made of the nodes left fixed by the isotropy group.

2 Symmetries in the bulk and graphs

When a conformal field theory has a symmetry group G , twisted boundary conditions around non-contractible loops may be considered [9]. They can be obtained from periodic boundary conditions by letting a group element act before closing the loop, or from the insertion of suitable disorder fields. The corresponding boundary conditions are labeled by group elements (there may be others [12]).

On a torus, there are two independent periods, chosen as 1 and τ . Choosing respectively g and e (i.e. periodic) boundary conditions along the two periods yields the partition function of the g -twisted sector in the form

$$Z_{g,e}(\tau) = \text{Tr}_{\mathcal{H}_g}[T_g] = \sum_{(j,j') \in \mathcal{H}_g} M_{j,j'}^{(g)} \chi_j^* \chi_{j'}, \quad (2.1)$$

where $T_g = q^{L_0 - \frac{c}{24}} \bar{q}^{\bar{L}_0 - \frac{c}{24}}$ is the (finite distance) transfer matrix with monodromy g . The twisted Hilbert space has been decomposed as $\mathcal{H}_g = \bigoplus_{(j,j')} M_{j,j'}^{(g)} \mathcal{R}_j \otimes \mathcal{R}_{j'}$ in terms of inequivalent representations of the symmetry algebra, with (integer) multiplicities $M_{j,j'}^{(g)}$.

A group element f maps \mathcal{H}_g to the isomorphic space $\mathcal{H}_{fgf^{-1}} \cong \mathcal{H}_g$ since the conjugation by f changes the monodromy of the transfer matrix:

$$Z_{g,e} = \text{Tr}_{\mathcal{H}_g}[f^{-1} f T_g f^{-1} f] = \text{Tr}_{\mathcal{H}_{fgf^{-1}}}[T_{fgf^{-1}}] = Z_{fgf^{-1},e}. \quad (2.2)$$

It follows that the centralizer of g has an action into \mathcal{H}_g . The symmetry means that the chiral algebra is left invariant by G , so that G can only rotate a whole representation to an equivalent one. Thus one finds

$$Z_{g,g'}(\tau) = \text{Tr}_{\mathcal{H}_g}[T_g g'] = \sum_{(j,j') \in \mathcal{H}_g} \lambda_{j,j'}(g; g') \chi_j^* \chi_{j'}, \quad gg' = g'g, \quad (2.3)$$

with $\lambda_{j,j'}(g; g')$ the character of g' acting on the $M_{j,j'}^{(g)}$ degenerate representations in the sector g . This action is unitary, and can be diagonalized, implying that all $\lambda_{j,j'}(g; g')$ can be written as a sum of roots of unity (charges), of order equal to the order of g' .

The twisted partition functions should not be intrinsically sensitive to a modular change of the modulus. Since a modular transformation mixes the two periods, the invariance holds

MIPF	A_n	$A_{4,2m+1}^*$	$A_{2m \geq 6}^*$	$D_6 = D_6^*$	$D_9 = D_9^*$	$D_{3m \geq 12}$	$D_{3m \pm 1}$	$D_{9 \neq n \geq 7}^*$	E_8	E_8^*
G	\mathbb{Z}_3	\mathbb{Z}_3	\mathbb{Z}_3	A_4, A_4'	$\mathbb{Z}_3, \mathbb{Z}_3'$	\mathbb{Z}_3	\mathbb{Z}_3	\mathbb{Z}_3	\mathbb{Z}_3	\mathbb{Z}_3
$\text{Aut } \Gamma^f$	\mathbb{Z}_3	–	\mathbb{Z}_2	$S_4, A_4 \times \mathbb{Z}_2$	S_3, \mathbb{Z}_3	S_3	–	\mathbb{Z}_3	\mathbb{Z}_6	\mathbb{Z}_2

cont'd	E_{12}	E_{12}^{MS}	E_{12}^{*MS}	E_{24}
	\mathbb{Z}_3	–	–	–
	$S_3, S_3, S_3 \times \mathbb{Z}_3$	–	$\mathbb{Z}_2 \times \mathbb{Z}_2$	\mathbb{Z}_2

Table 1: Groups of symmetry pertaining to the $su(3)$ affine models. The top line refers to the model (a modular invariant partition function), in a notation borrowed from [13]. The second line mentions the symmetry group G of the corresponding field theory. Two modular invariants (D_6 and D_9) allow two different realizations of the same symmetry group (A_4 and \mathbb{Z}_3 resp.). The third line gives the automorphism group(s) of the generating graph(s) associated with it.

provided the boundary conditions are accordingly changed. This results in the following identities

$$Z_{g,g'}(\tau) = Z_{g,gg'}(\tau + 1) = Z_{g'^{-1},g}(-\frac{1}{\tau}). \quad (2.4)$$

The presence of a symmetry group can be asserted by finding a consistent set of functions of the form (2.3) that transform covariantly under the modular group as in (2.4). This set of functions contains in particular the modular invariant partition function $Z_{e,e}$. For the affine models based on $su(2)$ and $su(3)$, this analysis has been completed in [10].

For the $su(2)$ affine models, classified by a list of ADE partition functions, the symmetry group G is the group of automorphisms of the Dynkin diagram A, D or E, associated with the modular invariant, except for the A_{n-1} models, n odd (that is, the diagonal theories for $\widehat{su}(2)_k$, k odd), which have no symmetry¹. For the $su(3)$ models, the symmetry groups are listed in Table 1 for all modular invariant partition functions.

Beyond the mere knowledge of the group, the partition functions must be given. We will not reproduce the list here, which can be found in [10], but we simply recall the relations they bear with graphs. The observations we made regarding these relations are summarized on the left half of Table 2, and will play an essential role on the cylinder.

Graphs were originally associated with a given modular invariant [7], more precisely with its diagonal terms:

$$Z_{e,e}(\tau) = \sum_{(j,j) \in \mathcal{H}_e} \lambda_{j,j}(e; e) |\chi_j(\tau)|^2 + \text{non-diagonal}. \quad (2.5)$$

¹In those cases, there is a symmetry \mathbb{Z}_2 but it is realized projectively [10].

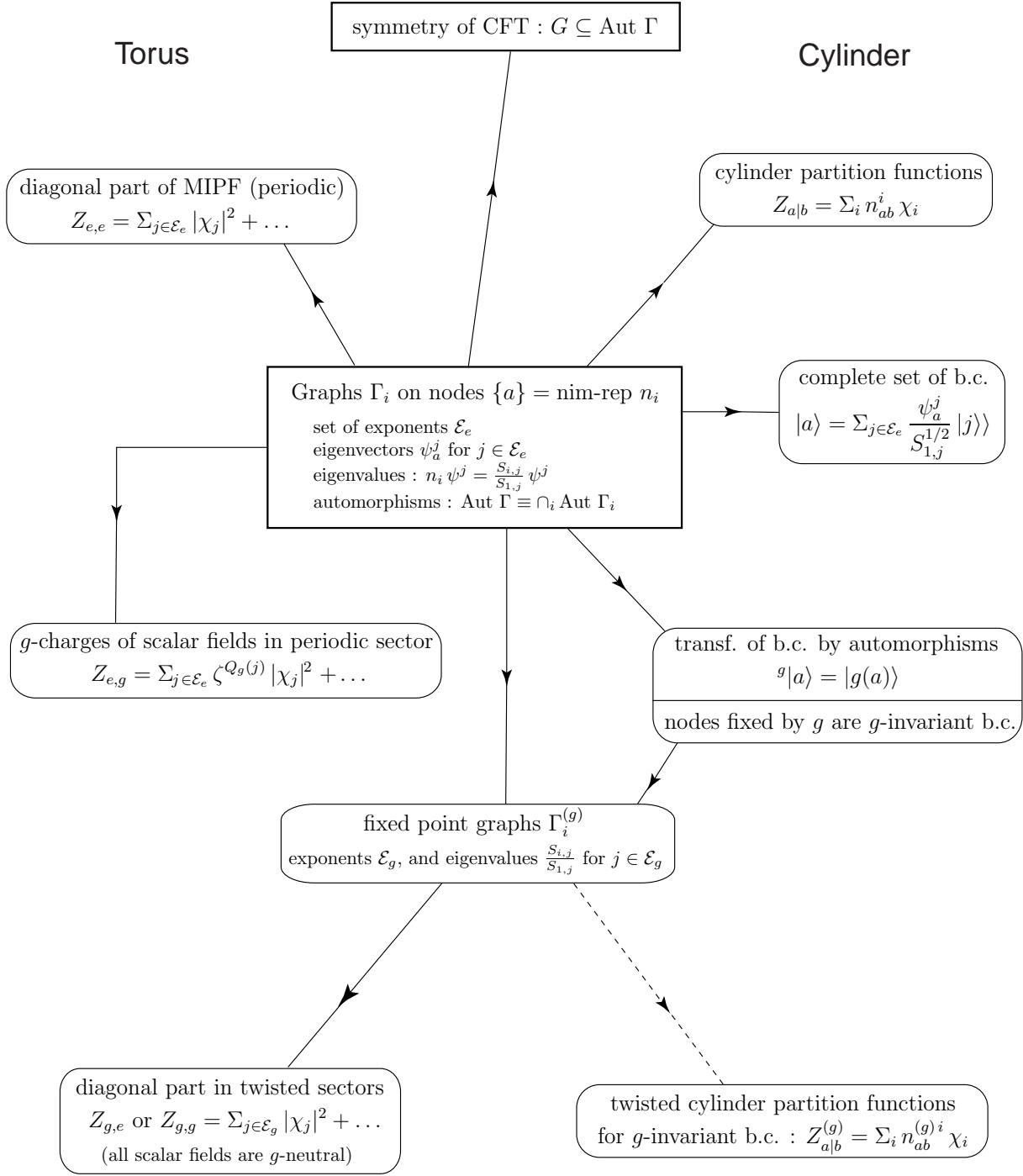


Table 2: Synoptic table summarizing the links that can be expected between the graphs and the various field theoretic data pertaining to an internal symmetry, as suggested by our analysis of Virasoro minimal models, and $su(2)$ and $su(3)$ affines models. All links are generically valid, although they are not systematic, in the sense that specific prescriptions are sometimes needed. In this respect, the arrow going from a fixed point graph to the twisted cylinder partition functions requires specific prescriptions (to produce the phased adjacency matrix $n^{(g)f}$ mentioned in the text), and for that reason, has been dashed.

One first defines the set of exponents $\mathcal{E}_e = \{j : (j, j) \in \mathcal{H}_e\}$, where j is taken in \mathcal{E}_e with multiplicity $\lambda_{j,j}(e; e)$. Thus the cardinal of \mathcal{E}_e is the number of diagonal terms in $Z_{e,e}$. To this set \mathcal{E}_e one then associates a collection of graphs $\{\Gamma^i\}$ such that the eigenvalues of the adjacency matrix n^i of Γ^i form the set $\{\frac{S_{i,j}}{S_{1,j}}\}$ for j in \mathcal{E}_e , with S the matrix representing the modular transformation $\tau \rightarrow -\frac{1}{\tau}$ on the characters. The number of graphs is equal to the dimension of the chiral fusion ring, each graph living on a set of $|\mathcal{E}_e|$ nodes, denoted $\{a\}$. From their spectrum, the matrices n^i form a non-negative integer representation of the fusion ring (a “nimrep”). Among them one can choose a set of generators, and the corresponding graphs, which we will call the generating graphs. For the $su(2)$ and $su(3)$ affine models, the fusion ring has one generator n^f , so that a single graph Γ^f needs be given for each modular invariant (the index f can be chosen to be a fundamental representation).

For $su(2)$, there is a unique generating graph associated with a modular invariant of A, D or E type, and it is precisely the corresponding Dynkin diagram [13]. For $su(3)$, one generating graph has been found for each modular invariant, but in three cases, several (isospectral) graphs are known [7]: the two self-conjugate invariants $D_6 = D_6^*$ and $D_9 = D_9^*$ can each be associated two different generating graphs, usually denoted by $\mathcal{D}_6, \mathcal{D}_6^*$ and $\mathcal{D}_9, \mathcal{D}_9^*$, while for the exceptional invariant E_{12} , three isospectral graphs $\mathcal{E}_{12}^{(1)}, \mathcal{E}_{12}^{(2)}, \mathcal{E}_{12}^{(3)}$ exist. The resulting list of graphs, likely to be complete but not proved to be so, can be found in [13].

It turns out that the symmetry group G in the affine models is very close to the automorphism group of the generating² graph Γ^f . The two groups are equal in all $su(2)$ models except those of even rank A type. For $su(3)$, two types of graphs are to be distinguished. The fold graphs³ associated to the theories $A_n^*, D_{3m\pm 1}$ and E_8^* have an automorphism group (often trivial) with no direct relation to G . For all other graphs, G is either equal to or is a subgroup of $\text{Aut } \Gamma^f$. (See Table 1 for a comparison of G and $\text{Aut } \Gamma^f$ in all cases.)

The relevance of the graph is strengthened by the following two observations [11, 10].

1. For the non-fold graphs, G is a subgroup of the graph automorphisms. As such it acts on an eigenbasis ψ^j of the adjacency matrices n^i . It leaves invariant the various eigensubspaces, labeled by the exponents of \mathcal{E}_e which are different, and so it acts by a block diagonal representation $R(g) = \bigoplus_{j:(j,j) \in \mathcal{H}_e} R_j(g)$, where R_j has dimension $M_{j,j}^{(e)} = \lambda_{j,j}(e; e)$ (the multiplicity of j in \mathcal{E}_e). Being the symmetry group of the conformal theory, G also acts on the periodic Hilbert space, in particular on the diagonal representations of \mathcal{H}_e . This action is again block diagonal since G can only mix degenerate $\mathcal{R}_j \otimes \mathcal{R}_j$, also labeled by the distinct exponents. Thus G acts on $(\mathcal{H}_e)_{\text{diag}}$ through a reduced representation $R'(g) = \bigoplus_{j:(j,j) \in \mathcal{H}_e} R'_j(g)$ of the same form as $R(g)$.

Our first observation is that $R'(g)$ coincides⁴ with $R(g)$ (up to the choice of basis,

²When there is more than one generator, one should consider instead the symmetries σ of the nimrep $n_{ab}^i = n_{\sigma(a), \sigma(b)}^i$, or equivalently the intersection $\bigcap_i \text{Aut } \Gamma^i$.

³They are graphs obtained by quotienting other graphs by automorphisms that act freely. The graphs mentioned in the text are the fold graphs of respectively $\mathcal{D}_n^*, \mathcal{A}_{3m\pm 1}$ and \mathcal{E}_8 by a \mathbb{Z}_3 group.

⁴This is a minor point but in fact, one of the two is going to be an anti-representation (the transpose

so they are equivalent). Therefore the numbers $\lambda_{j,j}(e; g)$ that appear in front of the diagonal terms of $Z_{e,g}$ are exactly the characters of the automorphism g acting on the eigenspaces of the matrices (graphs) n^i

$$Z_{e,g} = \sum_{j: (j,j) \in \mathcal{H}_e} \lambda_{j,j}(e; g) |\chi_j|^2 + \dots \quad \Leftrightarrow \quad \lambda_{j,j}(e; g) = \text{ch } R_j(g). \quad (2.6)$$

Therefore the G -charges of the scalar fields in the periodic sector are entirely dictated by the graph (the charges are noted $\zeta^{Q_g(j)}$ in Table 2; they are the eigenvalues of the representation $R_j(g)$ or $R'_j(g)$ of which $\lambda_{j,j}(e; g)$ is the character).

For the fold graphs of $su(3)$, it turns out that all scalar fields in $Z_{e,e}$ are invariant under the action of G (all $\lambda_{j,j}(e; g) = \lambda_{j,j}(e; e)$ for all g). Then the above observation still holds if we view the whole of G as having no action at all on the graphs. The same view will be suggested when going to the cylinder.

2. It is very natural to define sets of twisted exponents $\mathcal{E}_g = \{j : (j, j) \in \mathcal{H}_g\}$, where each j is again taken with the multiplicity of (j, j) in \mathcal{H}_g , and for each g , to look for matrices $n^{(g)i}$ whose spectrum is given by the set $\{\frac{S_{i,j}}{S_{1,j}}\}$ for j in \mathcal{E}_g . Clearly, the $n^{(g)i}$ again satisfy the fusion algebra and are polynomially expressible in terms of a few generators, in our case, just one, namely $n^{(g)f}$. It no longer is a positive integer matrix in general, but when its entries are sums of roots of unity, as will soon be the case, one can see it as the adjacency matrix of a phased graph.

A simple inspection of the torus twisted partition functions reveals the striking fact that $|\mathcal{E}_g|$ is equal to the number of graph nodes that are fixed by g

$$|\mathcal{E}_g| = \#\{a : g(a) = a\}. \quad (2.7)$$

Moreover, in many cases, the restriction $\Gamma^f|_g$ of the generating graph Γ^f to the nodes fixed by g has the right spectrum $\{\frac{S_{i,j}}{S_{1,j}}\}_{j \in \mathcal{E}_g}$. This is quite non-trivial in view of the wild spectral changes that a restriction usually causes. In those happy cases, one could set $n^{(g)f} = n^f|_g$ and generate all $n^{(g)i}$ from it⁵. Finally, it was noted that for $g \neq e$, all diagonal terms of $Z_{g,e}$ are invariant under the centralizer of g , except for the D_6 and D_6^* invariants of $\widehat{su}(3)_3$ (which are the only cases where the centralizer of g is not generated by g).

The second observation, which extends the first one to the non periodic sectors, is very suggestive, but as such, remains vague. In a sense it is the purpose of this article to give them a full and precise meaning on the cylinder. As we shall see, these observations will have very strong consequences on the determination of the cylinder partition functions.

of a representation), see Eq. (4.1). On the other hand, R' is known only through its character (from the partition functions).

⁵In these cases, $n^{(g)f}$ is still positive integer-valued but the other matrices $n^{(g)i}$ will not be positive (albeit integral).

3 Boundary conditions

The previous section spelt out the consequences of an internal symmetry for the formulation on a torus, and its connection to graphs. The question we ask now is what this analysis becomes on a cylinder.

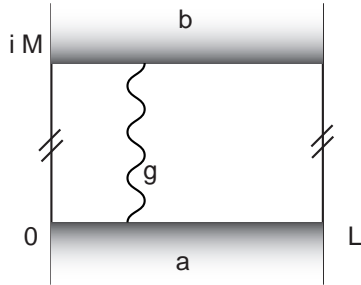


Figure 1: Geometric setting for the calculation of the cylinder twisted partition function $Z_{b|a}^{(g)}$.

We view the cylinder of perimeter L and length M as a rectangle, with vertical sides identified. There are two boundaries, the two horizontal sides, on which boundary conditions a and b are prescribed. We will require that they preserve the chiral algebra present, here an affine algebra. The way boundary conditions can be handled is by now standard [3, 4, 5, 13]; we will adapt the usual presentation in order to include the symmetry.

The symmetry group acts on all the states. In particular, it acts on the boundary conditions $a \rightarrow {}^g a$, and on the states that can circulate the non-trivial cycle. The latter may have a non-trivial monodromy around that loop, which, as before, can be implemented by the insertion of a disorder line, running from one boundary to the other (see Figure 1).

As usual, there are two equivalent ways of computing the partition function corresponding to boundary conditions a, b and monodromy g . If constant time slices are the closed horizontal circles, the proper Hilbert space to consider is \mathcal{H}_g , encountered above on the torus (and thus known). On the boundaries, the boundary conditions prescribe boundary states $|a\rangle$ and $|b\rangle$. Then the partition function is

$$Z_{b|a}^{(g)} = \langle b | e^{-MH_g} | a \rangle = \langle b | e^{-\frac{2\pi M}{L}(L_0 + \bar{L}_0 - \frac{c}{12})} | a \rangle. \quad (3.1)$$

If instead the time is chosen to run along the horizontal coordinate, the space corresponds to the segment of length M , with boundary conditions a and b at the two ends. The Hilbert space is now some $\mathcal{H}_{b|a}$ (independent of g). The time translation is periodic with period L , resulting in the partition function

$$Z_{b|a}^{(g)} = \text{Tr}_{\mathcal{H}_{b|a}} [g e^{-LH_{b|a}}] = \text{Tr}_{\mathcal{H}_{b|a}} [g e^{-\frac{\pi L}{M}(L_0 - \frac{c}{24})}]. \quad (3.2)$$

Various constraints on a, b and $\mathcal{H}_{b|a}$ arise from the equality of these two forms (moreover g may be varied independently over the symmetry group G).

The first form (3.1) shows that both boundary states a and b should have a projection in the space \mathcal{H}_g . If the projection of either one vanishes, so does the partition function $Z_{b|a}^{(g)}$. In the second form (3.2), the states of $\mathcal{H}_{b|a}$ are transported around the perimeter, acted on by g , and projected on themselves. The action of g turns the states of $\mathcal{H}_{b|a}$ into states of $\mathcal{H}_{g b|g a}$, so that the trace gives zero if the two boundary conditions a and b are not invariant under g (the form (3.2) would not be well-defined and would depend on the position of the disorder line).

Combining the two observations, one learns that boundary states belong to the Hilbert spaces \mathcal{H}_g for those g which leave them invariant :

$$|{}^g a\rangle = |a\rangle \iff |a\rangle \in \mathcal{H}_g. \quad (3.3)$$

The boundary states preserving the chiral algebra must satisfy the linear equations $[L_n - \bar{L}_{-n}]|a\rangle = 0$ for all $n \in \mathbb{Z}$, for conformal invariance, and similar equations with the affine generators J_n^c for the affine invariance. They form a vector space, where a convenient basis is provided by the so-called Ishibashi states $|j\rangle\rangle$. Each such state $|j\rangle\rangle$ satisfies the linear equations and belongs entirely to the diagonal representation $\mathcal{R}_j \otimes \mathcal{R}_j$, and vice-versa each diagonal representation $\mathcal{R}_j \otimes \mathcal{R}_j$ contains exactly one Ishibashi state $|j\rangle\rangle$. These solutions form a complete set of solutions (over \mathbb{C}).

The boundary conditions are required to belong to various Hilbert spaces \mathcal{H}_g , according to the above discussion. In each \mathcal{H}_g (they are all known), a basis is provided by the Ishibashi states belonging to that particular space, and we denote them by $|j\rangle\rangle_g$. They are labeled by the diagonal representations occurring in \mathcal{H}_g , that is, by the set called \mathcal{E}_g in the previous section (the diagonal terms of $Z_{g,e}$). The way these states are constructed shows in addition that

$${}_g \langle\langle j | q^{\frac{1}{2}(L_0 + \bar{L}_0 - \frac{c}{12})} | j' \rangle\rangle_{g'} = \chi_j(q) \delta_{j,j'} \delta_{g,g'}. \quad (3.4)$$

Consequently a boundary state invariant under a set of group elements g possesses many alternative writings, since for every such g , it can be written as a linear combination of Ishibashi states from the g -twisted sector, conventionally written with a factor of $S_{1,j}$,

$$|a\rangle = \sum_{j \in \mathcal{E}_g} \frac{\psi_a^{(g)j}}{\sqrt{S_{1,j}}} |j\rangle\rangle_g, \quad (3.5)$$

for some complex coefficients $\psi_a^{(g)j}$ to be determined.

The previous two equations allow to compute the partition function in the first form (3.1):

$$Z_{b|a}^{(g)} = \sum_{j \in \mathcal{E}_g} \frac{\psi_a^{(g)j} \psi_b^{(g)j*}}{S_{1,j}} \chi_j(\tilde{q}), \quad \tilde{q} = e^{-\frac{4\pi M}{L}}. \quad (3.6)$$

In order to compute the second form of it, one decomposes the space $\mathcal{H}_{b|a}$ into inequivalent representations of (a single copy of) the chiral algebra, $\mathcal{H}_{b|a} = \bigoplus_i n_{ba}^i \mathcal{R}_i$, with n_{ba}^i integers. If g leaves a and b invariant, it also commutes with the Hamiltonian $H_{b|a}$, and acts unitarily in $\mathcal{H}_{b|a}$ by rotating the equivalent representations, as before. Denoting by $n_{ba}^{(g)i}$ the character of the representation $L_{b|a}^i$ through which g acts on the n_{ba}^i equivalent representations ($n_{ba}^i = n_{ba}^{(e)i}$), one simply has from (3.2)

$$Z_{b|a}^{(g)} = \sum_i n_{ba}^{(g)i} \chi_i(q), \quad q = e^{-\frac{\pi L}{M}}. \quad (3.7)$$

Writing $\tilde{q} = e^{2i\pi\tilde{\tau}}$ and $q = e^{2i\pi\tau}$, the numbers $\tilde{\tau}$ and τ are related by a modular transformation. Transforming the partition function (3.6) in the basis of characters $\chi_i(q)$, one finds⁶ the Cardy equation, generalized to account for the symmetries,

$$(n_{ab}^{(g)i})^* = \sum_{j \in \mathcal{E}_g} \psi_a^{(g)j} \frac{S_{i,j}}{S_{1,j}} \psi_b^{(g)j*}, \quad \text{for } {}^g a = a, {}^g b = b. \quad (3.8)$$

The matrices satisfy $n^{(g)i^*} = (n^{(g)i})^\dagger$ for pairs i, i^* of conjugate fields. The other relation $n^{(g)i} = (n^{(g^*)i})^*$ follows if there is an involution $g \rightarrow g^*$, $a \rightarrow a^*$ such that $\mathcal{E}_g^* = \mathcal{E}_{g^*}$ and $(\psi_a^{(g)j})^* = \psi_{a^*}^{(g^*)j^*}$. The involution is of course trivial for $su(2)$. One checks that it also is trivial for the $su(3)$, except for the models A_n^* , D_n and E_8^* where $g^* = g^{-1}$ (g of order 3). Whenever $g^* \neq g$, the set of numbers $\{\frac{S_{i,j}}{S_{1,j}}\}_{j \in \mathcal{E}_g}$ is not closed under complex conjugation.

By definition, the boundary conditions are those combinations of Ishibashi states such that all the numbers n_{ab}^i are non-negative integers. As positive integral combinations of boundary conditions are again boundary conditions, it is convenient to find a set of extremal, *pure* ones (in analogy with the extremal equilibrium states), of which the others are positive superpositions. A usual though somewhat strong working hypothesis is to assume the existence of an orthonormal and complete set of pure boundary conditions, satisfying

$$\sum_{j \in \mathcal{E}_e} \psi_a^{(e)j*} \psi_b^{(e)j} = \delta_{a,b}, \quad \sum_a \psi_a^{(e)j*} \psi_a^{(e)j'} = \delta_{j,j'}. \quad (3.9)$$

It is by no means obvious that complete boundary conditions exist in all sensible rational (even unitary) conformal theories. Indeed infinitely many examples of affine (WZW) theories are known where an orthonormal and complete set does not exist [14]; they have however not been proved (or disproved) to be sensible theories in the first place. In a number of cases however, including the minimal Virasoro models, and the $su(2)$ and $su(3)$ models examined here, an orthonormal and complete set does exist.

⁶As is well-known, some care is needed. The identification of the coefficients of the characters relies on their linear independence, which, for current algebras, means using the unspecialized characters. This can be done by including sources (related to the Cartan subalgebra generators) in the stress-energy tensor, see the Appendix A in [13]. The explicit calculation in particular shows that the characters in (3.6) are related to those in (3.7) by an $S^{-1} = S^*$ transformation. This results in a complex conjugation in the Cardy equation.

The conditions (3.9) imply $n_{ab}^1 = \delta_{a,b}$ from Eq. (3.8), that is, the identity field occurs once in the space $\mathcal{H}_{a|a}$, and is absent from $\mathcal{H}_{b|a}$ for $a \neq b$. They also fix the number of pure boundary conditions to be $|\mathcal{E}_e|$.

More importantly, they show that the coefficients $\psi_a^{(e)j}$ are the entries of a unitary matrix. Thus from Eq. (3.8) for $g = e$, all matrices n^i are diagonalized by $\psi^{(e)}$ and have their spectrum given by the set $\{\frac{S_{i,j}}{S_{1,j}}\}_{j \in \mathcal{E}_e}$. Thus they form the nimreps discussed in the previous section, and can be seen as the adjacency matrices of graphs, the $\psi_a^{(e)j}$ being their common eigenvectors. The nodes of the graphs are thus seen to be labels for pure boundary conditions.

The completeness condition leads to a reformulation of the Cardy equation for $g = e$ as the problem of finding graphs with prescribed spectra [8]. Solving that problem yields an explicit determination of the boundary conditions themselves and all the partition functions $Z_{b|a} \equiv Z_{b|a}^{(e)}$. It remains to determine the functions $Z_{b|a}^{(g)}$ for the other g 's.

Before doing that, let us note that the partition functions have the expected symmetries $Z_{b|a} = Z_{g_b|g_a}$, valid for all boundary conditions and all g in G . This is easily seen from a double insertion of g and g^{-1} on two lines parallel to the perimeter of the cylinder, with say the g -line close to the a boundary, and the g^{-1} -line close to the b boundary. The two insertions amount to no effect at all, but each line can be pulled over to the closest boundary, transforming the boundary condition.

4 Symmetry of boundary conditions

For solving the Cardy equation for the other g 's, one first needs to determine the way a symmetry group element g acts on the (pure) boundary conditions. For doing that, the observations of Section 2 are essential (we emphasize that we have no proof that these hold in all generality).

The previous section gives us the boundary conditions as graph nodes and as elements of \mathcal{H}_e , i.e. as linear combinations of Ishibashi states $|j\rangle_e$ from the periodic sector. The diagonal terms of the torus partition function $Z_{e,g}$ say how these states transform under the symmetry group G , namely through the R' representation (see Section 2). The observation we made was that this action precisely coincides with the action of g , seen as an automorphism of the graphs, on the eigenvectors of the adjacency matrices n^i . Introducing a degeneracy index α_j taking values between 1 and the multiplicity of the exponent j , one easily deduces

$${}^g|a\rangle = \sum_{j \in \mathcal{E}_e} \frac{\psi_a^{(e)j}}{\sqrt{S_{1,j}}} {}^g|j\rangle_e = \sum_{j, \alpha_j, \alpha'_j} \frac{\psi_a^{(e)(j, \alpha_j)}}{\sqrt{S_{1,j}}} [R'_j(g)]_{\alpha_j, \alpha'_j} |j, \alpha'_j\rangle_e = \sum_{j \in \mathcal{E}_e} \frac{\psi_{g(a)}^{(e)j}}{\sqrt{S_{1,j}}} |j\rangle_e = |g(a)\rangle. \quad (4.1)$$

Thus the action of a symmetry g on the boundary conditions $\{a\}$ is simply the action of the corresponding automorphism on the graph nodes. In particular, the g -invariant boundary conditions correspond to the nodes that are left fixed by the automorphism associated to g . From Eq. (2.7), their number is equal to $|\mathcal{E}_g|$. We noted in Section 2 that for the $su(3)$

models associated to fold graphs (A_n^* , $D_{3m\pm 1}$ and E_8^*), all representations R'_j are trivial. It follows that all boundary conditions are invariant in those cases, and we can set $g(a) = a$ for all a .

All g -invariant boundary conditions are linear combinations of Ishibashi states taken from \mathcal{H}_g . The unknown coefficients $\psi_a^{(g)j}$ are constrained by the Cardy equation

$$(n_{ab}^{(g)i})^* = \sum_{j \in \mathcal{E}_g} \psi_a^{(g)j} \frac{S_{i,j}}{S_{1,j}} \psi_b^{(g)j*}. \quad (4.2)$$

As in the usual case $g = e$, a solution to this equation yields at the same time the coefficients $\psi^{(g)}$ and the matrices $n^{(g)i}$. There is however an additional global condition on these matrices, which expresses the fact that $n_{ab}^{(g)i}$ result from the action of a group element g .

Define $G_a \subset G$ to be the isotropy group of the boundary condition a . Then for $G_{a,b} = G_a \cap G_b$ the isotropy group of the pair a and b , the numbers $n_{ab}^{(g)i}$ for fixed i, a and b , must be the character of a representation of $G_{a,b}$. As discussed in the previous section, $G_{a,b}$ acts in the Hilbert space $\mathcal{H}_{a|b} = \oplus_i n_{ab}^i \mathcal{R}_i$, by a block diagonal representation $L_{a|b} = \oplus_i L_{a|b}^i$. Then, one must have ($n_{ab}^i = n_{ab}^{(e)i}$)

$$\forall i : \quad n_{ab}^{(g)i} = \text{ch } L_{a|b}^i(g), \quad \forall g \in G_{a,b}. \quad (4.3)$$

For the affine theories based on $su(2)$ and $su(3)$, the data pertaining to $g = e$ are assumed to be known: the matrices n^i are adjacency matrices of graphs, satisfying the fusion algebra, and $\psi^{(e)}$ is their diagonalizing matrix. Our task is to find the data for the other g 's satisfying the above conditions, Eqs. (4.2) and (4.3). This is done in the following section, and relies heavily on the same graphs that give the data for $g = e$.

Before that, we make a last comment regarding the transformation of the boundary conditions. That certain boundary conditions can be viewed as elements of different Hilbert spaces \mathcal{H}_g makes sense, as a boundary condition may be compatible with various monodromies. A concrete example is the free boundary condition in the Ising model, which is not more periodic than antiperiodic.

However their transformation under G , in Eq. (4.1), has been obtained from their expression as elements of \mathcal{H}_e . It showed which of them are invariant under which g 's, and in turn, this enabled us to write them as elements of other spaces \mathcal{H}_g . Consistency requires to make sure that these alternative writings are compatible with the transformations used to define them. Using once again the observations of Section 2, it is not difficult, but instructive, to see that they are indeed consistent (see however a peculiarity for the D_6 and D_6^* models of $su(3)$). The arguments are presented in an appendix.

5 Solutions to the twisted Cardy equation

The number of boundary conditions invariant under a given group element g is equal to $|\mathcal{E}_g|$, the number of diagonal terms in the twisted partition functions $Z_{g,e}$. An immediate consequence is that all matrices $\psi^{(g)}$ are square.

We have $n_{ab}^1 = \delta_{a,b}$ from the completeness condition. This implies $n_{ab}^{(g)1} = \delta_{a,b}$ for all g -invariant boundary conditions a, b , because it must be diagonal (compatibility with n_{ab}^1), with real positive entries (from Eq. (4.2)). It means that $\psi^{(g)}$ is unitary, and this in turn implies that all $(n^{(g)i})^*$ are diagonalizable, have a spectrum equal to the set of ratios $\{\frac{S_{i,j}}{S_{1,j}}\}_{j \in \mathcal{E}_g}$, and therefore form a representation of the fusion algebra. It also means that the g -invariant pure boundary conditions (nodes fixed by g) form a complete set of g -invariant boundary conditions.

This puts the twisted data $n^{(g)i}$ on an equal footing as the untwisted n^i , with of course several important differences. First, the $n^{(g)i}$ form representations of the fusion ring of (much) lower dimension than the n^i . Second, the matrices $n^{(g)i}$ for fixed g , do not yield a nimrep , but instead a $\mathbb{Z}(\zeta_N)$ -valued representation of the fusion algebra, with $\zeta_N = e^{\frac{2i\pi}{N}}$ and N the order of g . One may interpret them as adjacency matrices of phased graphs, in which links are assigned phases.

In addition, all $n^{(g)i}$ must be compatible with n^i , in the sense of Eq. (4.3). It implies, for every $g \in G_{a,b}$ (the subgroup leaving the boundary conditions a and b invariant), that

$$(n_{ab}^{(g)i})^* = n_{ab}^i \pmod{1 - \zeta_N}. \quad (5.1)$$

This compatibility condition can be put in yet another but more suggestive form: for fixed g , the matrix $(n^{(g)i})^*$, with indices in the set of g -invariant boundary conditions, is equal to the restriction of n^i to the nodes fixed by g , up to phases (N -th roots of unity).

This firmly suggests that solutions $(n^{(g)i})^*$ to the Cardy equation can be obtained from those subgraphs, by going through the following steps:

- (i) consider the subgraphs $\Gamma^{(g)i}$ corresponding to the nodes fixed by g and their adjacency matrices;
- (ii) weigh the adjacency matrices by appropriate phases so as to reproduce the correct spectra $\{\frac{S_{i,j}}{S_{1,j}}\}_{j \in \mathcal{E}_g}$ (that is, attach every link of the subgraphs a phase rather than the number 1, thus obtaining a phased graph);
- (iii) then check the compatibility of the resulting matrices with the n^i .

This procedure has been used to obtain the solutions $n^{(g)i}$ presented below in terms of the generator $n^{(g)f}$ of the fusion algebra ($n^{(g)f*} = (n^{(g)f})^\dagger$ for $su(3)$). It turns out that in $su(2)$, $n^{(g)f}$ is the plain restriction of n^f to the nodes fixed by g , i.e. no weight on the links. On the other hand, non trivial weights are required in $su(3)$ in order to achieve the desired spectra.

This procedure however does not guarantee the uniqueness of the solutions it produces. The generating matrices $n^{(g)f}$ are primarily constrained by their spectrum, so that a unitary redefinition $n^{(g)f} \rightarrow U n^{(g)f} U^\dagger$ is possible (it amounts to $\psi^{(g)} \rightarrow U \psi^{(g)}$), provided the entries of U are combinations of n -th roots of unity, so that the new matrices be still $\mathbb{Z}(\zeta_N)$ -valued.

The other constraints on $n^{(g)f}$ are the group properties with respect to g : their entries must be characters, Eq. (4.3). A general unitary rotation replaces $n_{ab}^{(g)f}$ by a linear combination $\sum_{a',b'} U_{aa'} U_{bb'}^* n_{a'b'}^{(g)f}$ taken over all sectors a', b' invariant under g . It is very implausible that these new combinations will have the character property. Moreover they mix characters from different sectors of boundary conditions, which is very unnatural (the various boundary conditions being mixed have also different isotropy groups in general).

Although we have no formal proof of that, the only reasonable unitary transformations are by diagonal matrices, which transform $n_{ab}^{(g)f}$ into $\varphi_a \varphi_b^* n_{ab}^{(g)f}$, for some roots of unity φ_a (which suitably depend on g so as to keep the character property). At least, we could verify in all models but the D_n series of $su(3)$ that no other transformation is allowed.

We will not dwell on this phase freedom. It has been discussed in the Virasoro minimal models in [11], where the Perron–Frobenius theorem was used to determine these phases, relying on the assumption that the non–degeneracy of the groundstate of the transfer matrix carried over in the continuum limit. The affine models however are not known to be the scaling limits of critical lattice models. Moreover, in $su(3)$ models, the groundstate turns out to be degenerate for many boundary conditions. Consequently there is no natural way to fix the freedom in the phases. (We note nonetheless that all solutions for $su(2)$ given below, satisfy the Perron–Frobenius criterion.)

5.1 The $su(2)$ twisted partition functions

The $su(2)$ models are particularly simple since the subgraphs of nodes fixed by a symmetry are very small, containing one or two nodes. Only the D_n theories have a large, unbounded number of fixed points.

The theories with a symmetry group are the series of A_{n-1} for n even, all $D_{\frac{n}{2}+1}$ for $n \geq 6$ and E_6 , which all have a \mathbb{Z}_2 symmetry, except the theory D_4 which has an S_3 symmetry. The subgraphs fixed by the \mathbb{Z}_2 form, respectively, an \mathcal{A}_1 graph, an $\mathcal{A}_{\frac{n}{2}-1}$ graph and an \mathcal{A}_2 graph. The D_4 theory has in addition a single node fixed by the \mathbb{Z}_3 (and by the whole of S_3).

In all cases, one may verify that the adjacency matrix of the subgraph $\Gamma^{(g)f}$ of nodes fixed by g provides a solution $n^{(g)f}$ to the Cardy equation and satisfies the group property (4.3). The other matrices $n^{(g)i} = U_{i-1}(n^{(g)f})$ are generated from $n^{(g)f}$ via the Tchebychev polynomials. They are in general different from the restriction of the n^i to the fixed subgraphs.

The simplest way to see that they have the right spectrum is to observe that all fixed subgraphs are $\mathcal{A}_{n'-1}$ graphs for a level $n' = \frac{n}{d}$ which divides the level n of the original theory, and that the set of twisted exponents is exactly equal to $\mathcal{E}_g = d \mathcal{E}_e(A_{n'-1})$ (see [10] for the sets \mathcal{E}_g).

5.2 The $su(3)$ twisted partition functions

We quickly review the results for the $su(3)$ models. We again distinguish two classes of theories. One class contains the models A_n^* , $D_{3m\pm 1}$ and E_8^* , which are those associated with a fold graph (unlike the $su(2)$ notations, the subscript refers here to the height $k + 3$). The other class to be discussed includes the models A_{3m} , D_{3m} , D_6^* and E_{12} . All other models either have no symmetry at all, or no invariant boundary conditions.

The main difference between the two classes is that the models in the first one have all their boundary conditions invariant under the symmetry. They are also more easily handled.

For $i = (a, b)$ the integrable highest weights, we denote the automorphisms (simple currents) of $\widehat{su}(3)_k$ as $\mu(a, b) = (n - a - b, a)$, with respect to which the S modular matrix transforms as $S_{\mu(i), j} = \omega^{t(j)} S_{i, j}$, where $t(j)$ is the triality of j and $\omega = e^{2i\pi/3}$.

In what follows, we will take the sets of exponents \mathcal{E}_g as given by the twisted partition functions computed in [10].

5.2.1 First class : the fold graphs

All models in this class have \mathbb{Z}_3 as symmetry group, and thus three sets of exponents \mathcal{E}_e , \mathcal{E}_g and \mathcal{E}_{g^2} . In all cases, these three sets have the same cardinality, and are explicitly given as $\mathcal{E}_{g^2} = \mathcal{E}_g^*$ and $\mathcal{E}_g = \mu^r(\mathcal{E}_e)$, with $r = 1$ or 2 depending on the models. For instance, for the $D_{3m\pm 1}$ theories, we find $\mathcal{E}_e = \{i : t(i) = 0\}$, $\mathcal{E}_g = \{i : t(i) = \mp 1\}$ and $\mathcal{E}_{g^2} = \{i : t(i) = \pm 1\}$, so that $\mathcal{E}_g = \mu^2(\mathcal{E}_e)$. The other cases have $\mathcal{E}_g = \mu(\mathcal{E}_e)$.

A direct consequence is that the spectra of $(n^{(g)^i})^*$ and n^i are related by

$$\left\{ \frac{S_{i,j}}{S_{1,j}} \right\}_{j \in \mathcal{E}_g} = \left\{ \frac{S_{i,\mu^r(j)}}{S_{1,\mu^r(j)}} \right\}_{j \in \mathcal{E}_e} = \omega^{rt(i)} \left\{ \frac{S_{i,j}}{S_{1,j}} \right\}_{j \in \mathcal{E}_e}, \quad (5.2)$$

so that $n^{(g)^i} = \omega^{-rt(i)} n^i$ is clearly a solution to the Cardy equation, with $\psi^{(g)} = \psi^{(e)}$. It is also manifestly compatible with n^i .

We note that, in the graphs associated to all these theories, some of the links are unoriented, which means that the corresponding sector of boundary conditions has a degenerate groundstate ($n_{ab}^f = n_{ab}^{f*}$).

5.2.2 Second class : the colourable graphs

One finds in this class the A_{3m} , $D_{3m \geq 9}$ and E_{12} theories, all with a \mathbb{Z}_3 symmetry, and also the D_6 and D_6^* theories, both with an A_4 symmetry.

(i) The series A_{3m} is trivial: the node $(m, m) = (\frac{n}{3}, \frac{n}{3})$ is the only one that is invariant under \mathbb{Z}_3 . Since $S_{f,(m,m)} = 0$, Cardy's equation says that $n^{(g)^f} = n^{(g^2)^f} = 0$ is the only solution (which coincides with the restriction of n^f to the fixed node).

(ii) The E_{12} theory has three nodes invariant under the \mathbb{Z}_3 , and the spectrum of $n^{(g)^f}$ is found to be $\{1, \omega, \omega^2\}$, the three third roots of unity. Three distinct graphs $\mathcal{E}_{12}^{(i)}$ have been

associated to the E_{12} modular invariant. Together, they yield two different solutions $n^{(g)f}$ to the Cardy equation:

$$\text{graph } \mathcal{E}_{12}^{(1)} : \quad n^f \Big|_{\text{fixed}} = \begin{pmatrix} 0 & 2 & 0 \\ 0 & 0 & 2 \\ 1 & 0 & 0 \end{pmatrix} \longrightarrow n^{(g)f} = \begin{pmatrix} 0 & -1 & 0 \\ 0 & 0 & -1 \\ 1 & 0 & 0 \end{pmatrix}, \quad (5.3)$$

$$\text{graphs } \mathcal{E}_{12}^{(2,3)} : \quad n^f \Big|_{\text{fixed}} = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix} \longrightarrow n^{(g)f} = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}. \quad (5.4)$$

For the first one, the matrix $n^{(g)f}$ corresponds to assigning the phases ω and ω^2 to the two double links.

(iii) In the case of D_6 and D_6^* , the group elements in A_4 have order 2 or order 3. The theory D_6^* has no invariant boundary condition under order 3 elements, while D_6 has, but is treated below, along with the whole series D_{3m} . For order 2 elements, the two theories D_6 and D_6^* behave in a similar way, and can be treated simultaneously.

The group A_4 has three elements of order 2, all conjugate to each other. The discussion is the same for any of them (only the nodes that are invariant change in D_6^*). Picking one particular g of order 2, we find two invariant boundary conditions (in the D_6 model, they are invariant under the full A_4). The set of twisted exponents is in both cases $\mathcal{E}_g = \{(2, 2), (2, 2)\}$, yielding a spectrum of $n^{(g)f}$ equal to $\{\frac{S_{f,j}}{S_{1,j}}\}_{j \in \mathcal{E}_g} = \{0, 0\}$. Cardy's equation implies $n^{(g)f} = 0$ identically, which is the restriction of n^f in the case of D_6^* . For D_6 , the restriction of n^f is equal to $\begin{pmatrix} 0 & 2 \\ 0 & 0 \end{pmatrix}$, and $n^{(g)f}$ simply corresponds to assigning the double link the phases $+1$ and -1 .

(iv) It remains to discuss the series D_{3m} , for which

$$\mathcal{E}_e = \{j : t(j) = 0\} \cup \{(m, m), (m, m)\}, \quad (5.5)$$

$$\mathcal{E}_g = \{j : t(j) = 2\}, \quad \mathcal{E}_{g^2} = \{j : t(j) = 1\} = \mathcal{E}_g^*. \quad (5.6)$$

Let us recall that the graph \mathcal{D}_{3m} is obtained from \mathcal{A}_{3m} by an orbifold procedure. Decomposing all weights i of the alcôve as $B_{3m} = T_0 \cup T_1 \cup T_2 \cup \{(m, m)\}$, with $T_k = \mu(T_{k-1})$, the orbifolding triplicates the μ -fixed point (m, m) , and establishes the identifications $i \sim \mu(i)$ in the three sets T_k . The geometric symmetry \mathbb{Z}_3 of the graph \mathcal{A}_{3m} is broken by these identifications, but is restored in \mathcal{D}_{3m} because the three nodes coming from the triplication of (m, m) are symmetrical (the graph has a symmetry S_3). The resulting graph \mathcal{D}_{3m} (its adjacency matrix) has a spectrum that is the zero triality subset of that of \mathcal{A}_{3m} , modulo the fact that (m, m) occurs three times.

All nodes of \mathcal{D}_{3m} except the triplicated ones are fixed points under the \mathbb{Z}_3 symmetry, and in number equal to any $|T_k|$. It is easy to see that a 'phased' orbifold procedure can be defined that instead selects the triality one or two subset of the spectrum of \mathcal{A}_{3m} , and leads to the phased graphs with the spectrum corresponding to the exponents \mathcal{E}_g or \mathcal{E}_{g^2} .

According to the above decomposition $B_{3m} = T_0 \cup T_1 \cup T_2 \cup \{(m, m)\}$, the adjacency matrix of the graph \mathcal{A}_{3m} can be written

$$\mathcal{A}_{3m} = \begin{pmatrix} A_0 & A_1 & A_2 & \alpha \\ A_2 & A_0 & A_1 & \alpha \\ A_1 & A_2 & A_0 & \alpha \\ \beta & \beta & \beta & 0 \end{pmatrix}, \quad (5.7)$$

where α and β are respectively column and row vectors. Since this graph encodes the fusion by the fundamental f , its eigenvalues are $\{\frac{S_{f,j}}{S_{1,j}}\}_{j \in B_n}$. With respect to the same decomposition, its eigenvectors $\psi_i^{(j)} = S_{j,i}$ can be written as $(v^{(j)}; \omega^{t(j)} v^{(j)}; \omega^{2t(j)} v^{(j)}; S_{j,(m,m)})$ with $v^{(j)} = (S_{j,i})_{i \in T_0}$. The eigenequation implies in particular

$$[A_0 + \omega^{t(j)} A_1 + \omega^{2t(j)} A_2] v^{(j)} + \alpha S_{j,(m,m)} = \frac{S_{f,j}}{S_{1,j}} v^{(j)}. \quad (5.8)$$

The restriction to those j of triality 2 shows that the matrix $A_0 + \omega^2 A_1 + \omega A_2$ has a spectrum equal to $\{\frac{S_{f,j}}{S_{1,j}}\}_{j \in \mathcal{E}_g}$. Similarly one finds that $A_0 + \omega A_1 + \omega^2 A_2$ has a spectrum equal to $\{\frac{S_{f,j}}{S_{1,j}}\}_{j \in \mathcal{E}_{g^2}}$. Thus the two matrices yield solutions to the twisted Cardy equation

$$n^{(g)} f = A_0 + \omega A_1 + \omega^2 A_2, \quad n^{(g^2)} f = (n^{(g)} f)^* = A_0 + \omega^2 A_1 + \omega A_2. \quad (5.9)$$

In terms of phased graphs, the solutions $n^{(g)} f$ and $n^{(g^2)} f$ involve a few phased links $(2m - 3)$, namely those which connect the sets T_0 and T_1 , as illustrated in Figure 2.

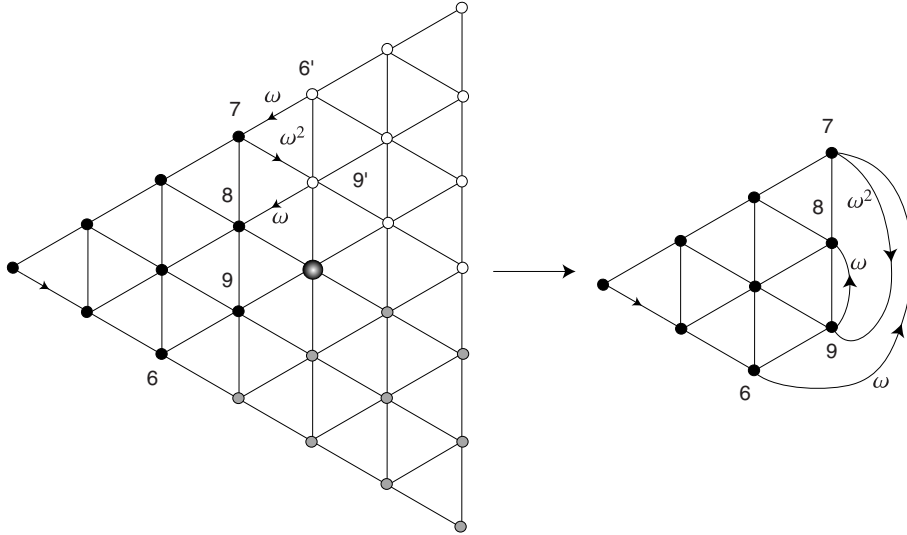


Figure 2: Shown on the right is the phased graph $(n^{(g)} f)^*$ for \mathcal{D}_9 , as resulting from a twisted orbifold of the graph \mathcal{A}_9 . The three colours differentiate the three domains T_k related by μ automorphisms, under which the central node is fixed. All the links are oriented.

A Boundary conditions and twisted Ishibashi states

A pure boundary condition a can be expanded in the basis of Ishibashi states from the periodic sector. We used this expansion to determine their transformation properties under the symmetry group G . We have then argued that a boundary condition that is invariant under a subgroup G_a of G can be expanded in Ishibashi states from all sectors twisted by elements of G_a :

$$|a\rangle = \sum_{j \in \mathcal{E}_g} \frac{\psi_a^{(g)j}}{\sqrt{S_{1,j}}} |j\rangle\rangle_g, \quad g \in G_a. \quad (\text{A.1})$$

For consistency, all these expressions for $g \neq e$ must be compatible with the transformations of a under G . We show that it is indeed the case (except in one instance). The arguments below apply equally well to the minimal Virasoro models.

We fix a and a g in G_a , and we consider the action of an arbitrary element f of G on the expression of $|a\rangle$ as element of \mathcal{H}_g , Eq. (A.1). We distinguish the two cases according to whether f is in the centralizer $C(g)$ of g or not.

An $f \notin C(g)$ transforms $|j\rangle\rangle_g$ into an Ishibashi state of $\mathcal{H}_{fgf^{-1}}$. Assuming no mixing (by a proper choice of basis), that is, $f|j\rangle\rangle_g = |j\rangle\rangle_{fgf^{-1}}$, we obtain

$$f|a\rangle = \sum_{j \in \mathcal{E}_g} \frac{\psi_a^{(g)j}}{\sqrt{S_{1,j}}} |j\rangle\rangle_{fgf^{-1}}. \quad (\text{A.2})$$

We note that the boundary state $|f(a)\rangle$ has isotropy group $G_{f(a)} = fG_a f^{-1}$, and can therefore be written as

$$|f(a)\rangle = \sum_{j \in \mathcal{E}_{fgf^{-1}}} \frac{\psi_{f(a)}^{(fgf^{-1})j}}{\sqrt{S_{1,j}}} |j\rangle\rangle_{fgf^{-1}}, \quad (\text{A.3})$$

for the same g . The Hilbert space $\mathcal{H}_{fgf^{-1}}$ is isomorphic to \mathcal{H}_g so that the set of exponents $\mathcal{E}_{fgf^{-1}}$ is equal to \mathcal{E}_g . Moreover the coefficients $\psi_{f(a)}^{(fgf^{-1})j}$ are related, through the Cardy equation, to the numbers $n_{f(a),f(b)}^{(fgf^{-1})i}$ specifying the (trace of the) action in the space $\mathcal{H}_{f(a)|f(b)}$ of the isotropy group $G_{f(a)} \cap G_{f(b)}$. This action is clearly conjugate to the action of $G_a \cap G_b$ in the Hilbert space $\mathcal{H}_{ab} \cong \mathcal{H}_{f(a)|f(b)}$. From this it follows that $n_{f(a),f(b)}^{(fgf^{-1})i} = n_{a,b}^{(g)i}$, and that the matrices $\psi^{(fgf^{-1})i}$ and $\psi^{(g)i}$ are equal (may be chosen to be equal). Thus $|f(a)\rangle$ can also be written as

$$|f(a)\rangle = \sum_{j \in \mathcal{E}_g} \frac{\psi_a^{(g)j}}{\sqrt{S_{1,j}}} |j\rangle\rangle_{fgf^{-1}}. \quad (\text{A.4})$$

Comparing (A.2) and (A.4), one sees that $f|a\rangle$ coincides with one of the possible writings of $|f(a)\rangle$. (Note that f may or may not be in G_a . A typical example is the D_4 invariant of $su(2)$, having $G = S_3$. The boundary condition corresponding to the middle node in the Dynkin diagram is fully invariant under S_3 : it can be written, in three different ways, as an element of the sector twisted by an order two element, and in two different ways, as

an element of the sector twisted by an order three group element. The \mathbb{Z}_2 expressions get permuted by the \mathbb{Z}_3 subgroup of S_3 , and vice-versa.)

If f is in the centralizer of g , it acts in \mathcal{H}_g (hence on the $|j\rangle_g$) in a way that can be read off from the torus partition function $Z_{g,f}$. It turns out, in all but two cases, that (i) the centralizer of g is the subgroup generated by g (hence f is a power of g), and (ii) all diagonal terms of $Z_{g,e}$ are invariant under g hence under f (i.e. $\lambda_{j,j}(g,f) = 1$ in the notations of Section 2). Together, these two facts imply that $|a\rangle$, written as an element of \mathcal{H}_g , is invariant under f .

The two cases where points (i) and (ii) above do not hold are the D_6 and D_6^* theories of $\widehat{su}(3)_3$, when g is an element of order 2 in $G = A_4$. The group A_4 has three elements of order 2, all conjugate, forming with the identity a subgroup⁷ $\mathbb{Z}_2 \times \mathbb{Z}_2 \equiv \{e, a, a', aa'\}$. On the torus, both models D_6 and D_6^* carry the same action of this subgroup (they are distinguished by the realization of the \mathbb{Z}_3 subgroup).

The centralizer of any order 2 element is equal to $\mathbb{Z}_2 \times \mathbb{Z}_2$. The partition functions in the sector twisted by a say, read [10] (those in the sectors a' or aa' are similar)

$$Z_{a,a^k a'^l} = [1 + (-1)^l] |\chi_{(2,2)}|^2 + (-1)^k \left\{ \chi_{(2,2)}^* [\chi_{(1,1)} + \chi_{(4,1)} + \chi_{(1,4)}] + (-1)^l \times \text{c.c.} \right\}, \quad (\text{A.5})$$

for $k, l = 0, 1$. Concentrating on the two degenerate, diagonal terms, one sees that they are invariant under a , but have opposite charges under a' .

This has the following consequence. From the graphs \mathcal{D}_6 and \mathcal{D}_6^* , one sees that in each case, there are two nodes invariant under a (in \mathcal{D}_6 , they are actually invariant under the full A_4). The corresponding two boundary states, call them 1 and 2, can thus be expressed as linear combination of the two Ishibashi states of the sector twisted by a ($S_{(1,1),(2,2)} = \frac{1}{2}$) :

$$|1\rangle = \sqrt{2} \left[\psi_1^{(a)(2,2)} |(2,2)\rangle_a + \psi_1^{(a)\widetilde{(2,2)}} |\widetilde{(2,2)}\rangle_a \right], \quad (\text{A.6})$$

$$|2\rangle = \sqrt{2} \left[\psi_2^{(a)(2,2)} |(2,2)\rangle_a + \psi_2^{(a)\widetilde{(2,2)}} |\widetilde{(2,2)}\rangle_a \right]. \quad (\text{A.7})$$

In D_6^* , the graph says that the two boundary states are interchanged under a' or aa' . Assuming without loss of generality that $|(2,2)\rangle_a$ and $|\widetilde{(2,2)}\rangle_a$ have respective charges $+1$ and -1 under a' (both are $+1$ under a), the matrix of coefficients $\psi^{(a)} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$ makes the transformations $(a' \text{ or } aa')|1\rangle = |2\rangle$ and $(a' \text{ or } aa')|2\rangle = |1\rangle$ manifest, and is unitary.

In D_6 on the contrary, the two boundary states are invariant under a' and aa' , so one would like to determine the coefficients to make these invariances manifest. However there is no way to do it with a unitary matrix $\psi^{(a)}$. The reason for this is unclear to us.

⁷The modular invariant itself can be viewed as the reduction of the diagonal invariant of $\widehat{so}(8)_1$, in which $\widehat{su}(3)_3$ is conformally embedded. The subgroup $\mathbb{Z}_2 \times \mathbb{Z}_2$ survives the embedding and corresponds to the group of simple currents of $\widehat{so}(8)$. In terms of $\widehat{so}(8)$ primary fields however, the twisted torus partition functions contain no diagonal term.

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