Exponential Mixing of the 2D Stochastic Navier-Stokes Dynamics [1]

J.Bricmont
UCL, Physique Théorique, B-1348, Louvain-la-Neuve, Belgium

A.Kupiainen, R.Lefevere
Helsinki University, Department of Mathematics,
P.O.Box 4, Helsinki 00014, Finland

Abstract

We consider the Navier-Stokes equation on a two dimensional torus with a random force which is white noise in time, and excites only a finite number of modes. The number of excited modes depends on the viscosity ν, and grows like ν^{-3} when ν goes to zero. We prove that this Markov process has a unique invariant measure and is exponentially mixing in time.

1 Introduction

Homogenous isotropic turbulence is often mathematically modelled by Navier Stokes equation subjected to an external stochastic driving force which is stationary in space and time and "large scale", which in particular means smooth in space. The status of the existence and uniqueness of solutions to the stochastic PDE parallels that of the deterministic one. In particular, in two dimensions, it holds under very general conditions.

However, for physical reasons, one is interested in the existence, uniqueness and properties of the stationary state of the resulting Markov process. While the existence of such a state follows with soft methods [10], uniqueness, i.e. ergodic and mixing properties of the process has been harder to establish. In a nonturbulent situation, i.e. with a sufficiently rough forcing this was established in [5] and for large viscosity in [8]. The first result for a smooth forcing was by Kuksin and Shirikyan [7] who considered a periodically kicked system with bounded kicks. In particular they could deal with the case where only a finite number of modes are excited by the noise (the number depends both on the viscosity and the size of the kicks). In [2], we proved uniqueness and exponential mixing for such a kicked system where the kicks have a Gaussian distribution, but we required that there be a nonzero noise for each mode. In this paper, we extend that analysis to the case where only finitely many modes are excited, and the forcing is white noise in time. An essential ingredient in our analysis is the Lyapunov-Schmidt type reduction introduced in [7], that allows to transform the original Markov process with infinite dimensional state space to a non-Markovian process with

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finite dimensional state space. We apply standard ideas of statistical mechanics (high temperature expansions) to this process to deduce mixing properties of the dynamics. While preparing this manuscript we received a preliminary draft [4] that claims similar results, using a somewhat more probabilistic approach. We thank these authors for communicating us their ideas, some of which helped us to simplify our arguments, especially in Section 8 below.

We consider the stochastic Navier-Stokes equation for the velocity field $u(t, x) \in \mathbf{R}^2$ defined on the torus $\mathbf{T} = (\mathbf{R}/2\pi\mathbf{Z})^2$:

$$du + ((u \cdot \nabla)u - \nu\nabla^2 u + \nabla p)dt = df \tag{1}$$

where $f(t, x)$ is a Wiener process with covariance

$$Ef_\alpha(t, x)f_\beta(t', y) = \min\{t, t'\}C_{\alpha\beta}(x - y) \tag{2}$$

and $C_{\alpha\beta}$ is a smooth function satisfying $\sum_\alpha \partial_\alpha C_{\alpha\beta} = 0$. Equation (1) is supplemented with the incompressibility condition $\nabla \cdot u = 0 = \nabla \cdot f$, and we will also assume that the averages over the torus vanish: $\int_{\mathbf{T}} u(0, x) = 0 = \int_{\mathbf{T}} f(t, x)$, which imply that $\int_{\mathbf{T}} u(t, x) = 0$ for all times t.

It is convenient to change to dimensionless variables so that ν becomes equal to one. This is achieved by setting $u(t, x) = \nu u'(\nu t, x)$. Then u' satisfies (1), (2) with ν replaced by 1, and C by

$$C' = \nu^{-3}C.$$

¿From now on, we work with such variables and drop the primes. The dimensionless control parameter in the problem is the (rescaled) energy injection rate $\frac{1}{2}\mathrm{tr}\, C'(0)$, customarily written as $(\mathrm{Re})^3$ where Re is the Reynolds number:

$$\mathrm{Re} = \epsilon^{\frac{1}{3}}\nu^{-1},$$

and $\epsilon = \frac{1}{2}\mathrm{tr}\, C(0)$ is the energy injection rate in the original units (for explanations of the terminology see [6]).

In two dimensions, the incompressibility condition can be conveniently solved by expressing the velocity field in terms of the vorticity $\omega = \partial_1 u_2 - \partial_2 u_1$. First (1) implies the transport equation

$$d\omega + ((u \cdot \nabla)\omega - \nabla^2\omega)dt = db, \tag{3}$$

where $b = \partial_1 f_2 - \partial_2 f_1$ has the covariance

$$Eb(t, x)b(t', y) = \min\{t, t'\}(2\pi)^{-1}\gamma(x - y)$$

with $\gamma = -2\pi\nu^{-3}\Delta\mathrm{tr}C$.

Next, going to the Fourier transform, $\omega_k(t) = \frac{1}{2\pi}\int_{\mathbf{T}} e^{ik\cdot x}\omega(t, x)dx$, with $k \in \mathbf{Z}^2$; we may express u as $u_k = i\frac{(-k_2,k_1)}{k^2}\omega_k$, and write the vorticity equation as

$$d\omega(t) = F(\omega(t))dt + db(t), \tag{4}$$

where the drift is given by

$$F(\omega)_k = -k^2\omega_k + \frac{1}{2\pi}\sum_{l\in\mathbf{Z}^2\setminus\{0,k\}} \frac{k_1 l_2 - l_1 k_2}{|l|^2}\omega_{k-l}\omega_l \tag{5}$$

and $\{b_k\}$ are Brownian motions with $\bar{b}_k = b_{-k}$ and

$$Eb_k(t)b_l(t') = \min\{t, t'\}\delta_{k,-l}\,\gamma_k.$$

The dimensionless control parameter for the vorticity equation is

$$R = \sum_{k\in\mathbf{Z}^2} \gamma_k = 2\pi\gamma(0) \tag{6}$$

which is proportional to the ω injection rate, and also to the third power of the Reynolds number. We will be interested in the turbulent region $R \to \infty$; therefore, we will always assume below, when it is convenient, that R is sufficiently large.

For turbulence one is interested in the properties of stationary state of the stochastic equation (4) in the case of *smooth* forcing (see [1] for some discussion of this issue) and, ideally, one would like to consider the case where one excites only a finite number of modes,

$$\gamma_k \neq 0\ ,\ k^2 \leq N,$$

with N of order of one. In this paper we assume that N scales as

$$N = \kappa R, \tag{7}$$

with κ an absolute constant fixed below. We take all the other $\gamma_k = 0$, although this condition can easily be relaxed. Let us denote the minimum of the covariance by

$$\rho = \min\{|\gamma_k| \mid |k|^2 \leq N\}.$$

Before stating our result, we need some definitions. Let P be the orthogonal projection in $H = L^2(\mathbf{T})$ to the subspace H_s of functions having zero Fourier components for $|k|^2 > N$. We will write

$$\omega = s + l$$

with $s = P\omega$, $l = (1-P)\omega$ (respectively, the small k and large k parts of ω). Denote also by H_l the complementary subspace (containing the nonzero components of l). H is our probability space, equipped with $\mathcal{B}$, the Borel σ-algebra.

The stochastic equation (4) gives rise to a Markov process $\omega(t)$ and we denote by $P^t(\omega, E)$ the transition probability of this process.

Our main result is the

Theorem. *The stochastic Navier-Stokes equation* (4) *defines a Markov process with state space* $(H, \mathcal{B})$ *and for all* $R < \infty$, $\rho > 0$ *it has a unique invariant measure* μ *there. Moreover,* $\forall \omega \in H$, *for all Borel sets* $E \in H_s$ *and for all bounded Hölder continuous functions* F *on* H_l, *we have,*

$$\left| \int P^t(\omega, d\omega') 1_E(s') F(l') - \int \mu(d\omega') 1_E(s') F(l') \right| \leq C(\omega) ||F||_\alpha e^{-mt} \tag{8}$$

where $m = m(R, \rho, \alpha) > 0$, $||F||_\alpha$ *is the Hölder norm of exponent* α, *and* $C(\omega)$ *is a.s. finite.*

Remark 1. In a previous paper [1] we have shown that, with probability 1, the functions on the support of such a measure as constructed here are real analytic. In particular all correlation functions of the form

$$\int \mu(d\omega) \prod_i \nabla^{n_i} u(x_i)$$

exist.

Remark 2. The parameters in our problem are R and ρ. All constants that do not depend on them will be generically denoted by C or c. Besides, we write $C(X, Y, Z)$ for a "constant" depending only on X, Y, Z. These constants can vary from place to place, even in the same equation.

We close this section by giving the outline of the proof and explain its connection to ideas coming from Statistical Mechanics.

Let us start by observing that, if we neglect the nonlinear term in (4-5), we expect $\|\omega\|$ to be of order $R^{\frac{1}{2}}$, for typical realizations of the noise ($R^{\frac{1}{2}}$ is the typical size of the noise, and the $-k^2\omega_k$ term will dominate in eq. (4) for larger values of $\|\omega\|$). It turns out that similar probabilistic estimates hold for the full equation (4) as shown in Section 3. Now, if $\|\omega\|$ is of size $R^{\frac{1}{2}}$, the $-k^2\omega_k$

term will dominate the nonlinear term (which is roughly of size $\|\omega\|^2$) in eq. (4), for $|k| \geq \kappa R^{\frac{1}{2}}$, and one can expect that those modes (corresponding to l above) will behave somewhat like the solution of the heat equation and, in particular, that they will converge to a stationary state.

Thus, the first step is to express the l-modes in terms of the s-modes at previous times. This is done in Section 2 and produces a process for the s-modes that is no longer Markovian but has an infinite memory. In Statistical Mechanics, this would correspond to a system of unbounded spins (the s-modes) with infinite range interactions, with the added complications that, here, the measure is not given in a Gibbsian form, but only through a Girsanov formula, i.e. (18) below, and that time is continuous. Hence, we have to solve several problems: the possibility that ω be atypically large, the long range "interactions", and finally, showing that a version of the s-process with a suitable cutoff is ergodic and mixing.

The large ω problem is treated in Section 3, using probabilistic estimates developped in [1], which, in Statistical Mechanics, would be called stability estimates. The infinite memory problem is treated in Sections 4 and 5, which are inspired by the idea of "high temperature expansion" in Statistical Mechanics, namely writing the Gibbs measure or, here, the Girsanov factor, as sum of products of factors having a finite range memory and which become smaller as that range increases. However, in the situation considered here, carrying out this expansion requires a careful and non standard partition of the phase space (explained in Section 4). The problem is that, even though for typical noise, hence for typical ω's, the l-modes depend exponentially weakly on their past (see Section 2), thus producing, typically, "interactions" that decay exponentially fast, they may depend sensitively on their past when the noise is large. In the language of Statistical Mechanices, atypically large noise produces long range correlations.

This problem of sensitive dependence is coupled to the last problem, that of the convergence of the s-process with finite memory to a stationary state. We have to get lower bounds on transition probabilities and we can prove those (see Section 8) only when the s-modes remain for a sufficiently long time in a suitable region of the phase space; thus, if we did not control the sensitive dependence, we would not be able to carry out that last step. Finally, in Section 7, we prove the bounds on our "high temperature" expansion and, in Section 6, we use that expansion to prove the Theorem. Note that, because we deal with a stochastic process, we never have to "exponentiate" our expansion, unlike what one would usually has to do in Statistical Mechanics (i.e., the analogue of the partition function here equals 1). The choice of κ in (7) is explained in Remark 2 of Section 4.

2 Finite dimensional reduction

We will use an idea of [7] to reduce the problem of the study of a Markov process with infinite dimensional state space to that of a non-Markovian process with finite dimensional state space.

For this purpose, write the equation (4) for the small and large components of ω separately:

$$ds(t) \;=\; PF(s(t)+l(t))dt + db(t) \tag{9}$$

$$\tfrac{d}{dt}l(t) \;=\; (1-P)F(s(t)+l(t)). \tag{10}$$

The idea of [7] is to solve the l equation for a given function s, thereby defining $l(t)$ as a function of the entire history of $s(t')$, $t' \leq t$. Then the s equation will have a drift with memory. Let us fix some notation. For a time interval I we denote the restriction of ω (or s, l respectively) to I by $\omega(I)$, and use the boldface notation $\mathbf{s}(I)$, to constrast it with $s(t)$, the value of s at a single time. $\|\cdot\|$ will denote the L^2 norm. In [1] it was proven that, for any $\tau < \infty$, there exists a set $\mathcal{B}_\tau$ of Brownian paths $b \in C([0,\tau], H_s)$ of full measure such that, for $b \in \mathcal{B}_\tau$, (4) has a unique solution with $||\omega(t)|| < \infty$, $||\nabla\omega(t)|| < \infty$ for all t (actually, $\omega(t)$ is real analytic). In particular, the projections s and l of this solution are in $C([0,\tau], H_{s(l)})$ respectively.

On the other hand, let us denote, given any $\mathbf{s} \in C([0,\tau], H_s)$, the solution - whose existence we will prove below - of (10), with initial condition $l(0)$ by $l(t, \mathbf{s}([0,t]), l(0))$. More generally, given initial data $l(t')$ at time $t' < \tau$ and $\mathbf{s}([t',\tau])$, the solution of (10) is denoted, for $\sigma \le \tau$, by $l(\sigma, \mathbf{s}([t',\sigma]), l(t'))$ and the corresponding ω by $\omega(\sigma, \mathbf{s}([t',\sigma]), l(t'))$. The existence and key properties of those functions are given by:

Proposition 1. *Let $l(0) \in H_l$ and $s \in C([0,\tau], H_s)$. Then $l(\cdot, \mathbf{s}([0,t]), l(0)) \in C([0,\tau], H_l) \cap L^2([0,\tau], H_l^1)$, where $H_l^1 = H_l \cap H^1$, and H^1 is the first Sobolev space. In particular,*

$$\sup_{t\in[0,\tau]} \|l(t, \mathbf{s}([0,t]), l(0))\| \le C(R, \sup_{t\in[0,\tau]} \|s(t)\|, \|l(0)\|) \tag{11}$$

where the notation $C(R, \sup_{t\in[0,\tau]} \|s(t)\|, \|l(0)\|)$ is defined in Remark 2, Section 1. Moreover, given two initial conditions l_1, l_2 and $t \le \tau$

$$\|l(t, \mathbf{s}([0,t]), l_1) - l(t, \mathbf{s}([0,t]), l_2)\| \le \exp\left[-\kappa R t + a \int_0^t \|\nabla \omega_1\|^2\right] \|l_1 - l_2\| \tag{12}$$

where $a = (2\pi)^{-2} \sum |k|^{-4}$ and $\omega_1(t) = s(t) + l_1(t, \mathbf{s}([0,t]), l_1)$. The solution satisfies

$$l(t, \mathbf{s}([0,t]), l(0)) = l(t, \mathbf{s}([\tau,t]), l(\tau, \mathbf{s}([0,\tau]), l(0))). \tag{13}$$

Now, if $s = P\omega$ with ω as above being the solution of (4) with noise $b \in \mathcal{B}_\tau$ then the $l(s)$ constructed in the Proposition equals $(1-P)\omega$ and the stochastic process $s(t)$ satisfies the reduced equation

$$ds(t) = f(t)dt + db(t) \tag{14}$$

with

$$f(t) = PF(\omega(t)). \tag{15}$$

where $\omega(t)$ is the function on $C([0,t], H_s) \times H_l$ given by

$$\omega(t) = s(t) + l(t, \mathbf{s}([0,t]), l(0)) \tag{16}$$

(14) has almost surely bounded paths and we have a Girsanov representation for the transition probability of the ω-process in terms of the s-variables

$$P^t(\omega(0), F) = \int \mu^t_{\omega(0)}(d\mathbf{s}) F(\omega(t)) \tag{17}$$

with

$$\mu^t_{\omega(0)}(d\mathbf{s}) = e^{\int_0^t (f(\tau), \gamma^{-1}(ds(\tau) - \frac{1}{2} f(\tau) d\tau))} \nu^t_{s(0)}(d\mathbf{s}) \tag{18}$$

where $\nu^t_{s(0)}$ is the Wiener measure with covariance γ on paths $\mathbf{s} = \mathbf{s}([0,t])$ with starting point $s(0)$ and $(\cdot,\cdot)$ the ℓ^2 scalar product. We define the operator γ^{-1} in terms of its action on the Fourier coefficients:

$$(f, \gamma^{-1} f) = \sum_{|k|^2 \le N} |f_k|^2 \gamma_k^{-1}. \tag{19}$$

The Girsanov representation (17) is convenient since the problem of a stochastic PDE has been reduced to that of a stochastic process with finite dimensional state space. The drawback is that this process has infinite memory. In Sections 4 and 5 we present a formalism, borrowed from statistical mechanics, that allows us to approximate it by a process with finite memory; the approximation will be controlled in Section 7, while the finite memory process will be studied in Section 8. This analysis is mostly done in the s-picture, but an important ingredient in it will be some a priori estimates on the transition probabilities of the original Markov process generated by (4) that we prove in the next Section.

3 A priori estimates on the transition probabilities

The memory in the process (14) is coming from the dependence of the solution of (10) on its initial conditions. By Proposition 1, the dependence is weak if $\int_0^t \|\nabla\omega\|^2$ is less than cR for a suitable c. We localize the time intervals where this condition holds by inserting a suitable partition of unity in the expression (17). We shall show (in Section 8 below) that, during such time intervals, the s process behaves qualitatively like an ergodic Markov process. In this section we show that the complementary time intervals occur with small probability.

Let us first explain the partition of unity. We define, for each unit interval $[n-1,n] \equiv \mathbf{n}$, a quantity measuring the size of ω on that interval by:

$$D_n = \tfrac{1}{2}\sup_{t\in\mathbf{n}} ||\omega(t)||^2 + \int_{\mathbf{n}} ||\nabla\omega(t)||^2 dt. \tag{20}$$

Let $\{\phi_k\}_{k\in\mathbf{N}}$ be a smooth partition of unity for $\mathbf{R}^+$, with the support of ϕ_k contained in $[2^kR, 2^{k+2}R]$ for $k>0$, and in $[0,4R]$ for $k=0$. Set, for $\mathbf{k}\in\mathbf{N}^t$,

$$\chi_{\mathbf{k}}(\omega) = \prod_n \phi_{k_n}(D_n(\omega)). \tag{21}$$

We insert $1=\sum_{\mathbf{k}}\chi_{\mathbf{k}}$ in (18), to get

$$\mu^t_{\omega(0)}(d\mathbf{s}) = \sum_{\mathbf{k}} \chi_{\mathbf{k}}\mu^t_{\omega(0)}(d\mathbf{s}). \tag{22}$$

The following Proposition bounds the probability of the unlikely event that we are interested in:

Proposition 2. *There exist constants $c>0$, $c'<\infty$, $\beta_0<\infty$, such that for all t,t', $1\le t<t'$ and all $\beta\ge\beta_0$,*

$$P\Big(\sum_{n=t}^{t'-1} D_n(\omega) \ge \beta R|t'-t| \Big| \omega(0)\Big) \le \exp(\tfrac{1}{R}c'e^{-t}\|\omega(0)\|^2)\exp(-c\beta|t'-t|) \tag{23}$$

In order to prove Proposition 2, we need some Lemmas. We will start with a probabilistic analogue of the so-called enstrophy balance:

Lemma 3.1. *For all $\omega(0)\in L^2$, and all $t\ge 0$,*

$$E\Big[e^{\frac{1}{4R}\|\omega(t)\|^2} \Big| \omega(0)\Big] \le 3e^{\frac{1}{4R}e^{-t}\|\omega(0)\|^2}, \tag{24}$$

and

$$P(\|\omega(t)\|^2 \ge D|\omega(0)) \le 3e^{-\frac{D}{4R}}e^{\frac{1}{4R}e^{-t}\|\omega(0)\|^2} \tag{25}$$

Remark. This Lemma shows that the distribution of $\|\omega(t)\|^2$ satisfies an exponential bound on scale R with a prefactor whose dependence on the initial condition decays exponentially in time. Thus, if $\|\omega(0)\|^2$ is of order D, $\|\omega(t)\|^2$ will be, with large probability, of order R after a time of order $\log D$.

Proof. Let $x(\tau)=\lambda(\tau)\|\omega(\tau)\|^2 = \lambda(\tau)\sum_k|\omega_k|^2$ for $0\le\tau\le t$. Then by Ito's formula (remember that, by (6), $\sum_k\gamma_k = R$ and thus $\gamma_k\le R$, $\forall k$):

$$\begin{aligned}\tfrac{d}{d\tau}E[e^x] &= E[(\dot\lambda\lambda^{-1}x - 2\lambda\sum_k k^2|\omega_k|^2 + \lambda\sum_k\gamma_k + 2\lambda^2\sum_k\gamma_k|\omega_k|^2)e^x] \\ &\le E[((\dot\lambda\lambda^{-1} - 2 + 2\lambda R)x + \lambda R)e^x]\end{aligned} \tag{26}$$

where E denotes the conditional expectation, given $\omega(0)$, and where we used the Navier-Stokes equation (3), $|k| \geq 1$ for $\omega_k \neq 0$, and the fact that the nonlinear term does not contribute (using integration by parts and $\nabla \cdot u = 0$). Take now $\lambda(\tau) = \frac{1}{4R} e^{(\tau - t)}$ so that $\lambda \leq \frac{1}{4R}$, $\dot{\lambda}\lambda^{-1} = 1$, $\dot{\lambda}\lambda^{-1} - 2 + 2\lambda R \leq -\frac{1}{2}$ and $\lambda R \leq \frac{1}{4}$. So,

$$\tfrac{d}{d\tau} E[e^x] \leq E[(\frac{1}{4} - \frac{1}{2}x)e^x] \leq \frac{1}{2} - \frac{1}{4}E[e^x]$$

where the last inequality follows by using $(1-2x)e^x \leq 2 - e^x$. Thus, Gronwall's inequality implies that:

$$E[e^{x(\tau)}] \leq e^{-\frac{\tau}{4}} e^{x(0)} + 2 \leq 3e^{x(0)}$$

i.e., using the definition of $\lambda(\tau)$,

$$E\Big[\exp(\frac{e^{\tau - t}}{4R} \|\omega(\tau)\|^2)\Big] \leq 3 \exp(\frac{e^{-t}\|\omega(0)\|^2}{4R}),$$

This proves (24) by putting $\tau = t$; (25) follows from (24) by Chebychev's inequality. $\square$

4 Partition of the path space

Consider the expression (22) for the measure μ. Given $\mathbf{k}$, we will now decompose the time axis into regions where the equation (10) may have sensitive dependence on initial conditions and the complement of those regions. Motivated by Proposition 2, let us consider, for time intervals L, the expressions

$$\gamma_L = \sum_{\mathbf{n} \subset L} 2^{k_n}. \tag{27}$$

Let T be a number to be fixed later (in Sections 6-8), depending on ρ, the minimum of the noise covariance. Define

$$\beta(L) = \begin{cases} \beta |L| & \text{if } |L| > \frac{1}{2}T \\ \frac{1}{2}\beta T & \text{if } |L| \leq \frac{1}{2}T \end{cases} \tag{28}$$

β is a constant to be fixed later (see Remark 2 below). Call the time intervals with end points on the lattice $T\mathbf{Z}$ T-intervals, and, for an interval $L = [m, n]$, let $\bar{L}$ be the smallest T-interval containing $[m, n]$. Consider the set $\mathcal{L}$ of intervals L such that, either

$$\gamma_L > \beta(L), \tag{29}$$

or $L = [(n-1)T, nT]$, so that

$$2^{k_{nT}} > \beta' T, \tag{30}$$

where $\beta' < \beta$ is a constant also to be fixed later (see Remark 2 below). Let $\bar{\mathcal{L}}$ be the union of all $\bar{L}$ with $L \in \mathcal{L}$. We call the connected components of $\bar{\mathcal{L}}$ *large* intervals and the T-intervals of length T in its complement *small* intervals. Note that intervals of length T can be either small or large (those of length at least $2T$ are always large). Hence, we introduce *labels* small/large on those intervals. By construction, two large intervals are always separated by at least one small one.

Remark 1. "Large" and "small" refer to $\omega(J)$ being large or small, not to the size of the interval. We use this slightly misleading terminology for the sake of brevity. γ_L are the natural random variables entering in the sensitive dependence estimate (12) and whose probability distribution was studied in Proposition 2. Since the estimate (23) involves the initial condition at the beginning of the time interval we consider and, since this initial condition is the size of ω at the end of a time interval where (29) is violated, we need to be sure it does not dominate the bound (23). For that reason, we include in our set of unlikely events also the ones defined by (30).

Remark 2. The three constants in our construction, κ, β, β' entering (7), (29) and (30) are fixed as follows: $\beta' \geq \beta'_0$, $\beta \geq \beta(\beta')$ and $\kappa \geq \kappa(\beta)$.

Remark 3. The virtues of this partition of phase space can be seen in Lemma 4.1 and 7.4 below. The bound (**??**) and Proposition 2 will imply that large intervals are unprobable. On the other hand, (**??**) and (**??**) will allow us to show that the argument of the exponential in (12) is less than $-cRT$, when the interval $[0,t]$ is replaced by an interval strictly including one of the intervals constructed here. This property will be essential in order to obtain bounds on the terms of the expansion constructed in the next Section.

Taken together, the small and large intervals form a partition $\pi(\mathbf{k}) = J_1, \ldots, J_N$ of the total time interval $[0,t]$. We arrange them in temporal order and write $J_i = [\tau_{i-1}, \tau_i]$ with $\tau_0 = 0$, $\tau_N = t$.

In order to obtain the analogue of what in Statistical Mechanics is called the high temperature expansion, we need to write the sum in (22) as a sum of products of independent factors. As a first step in that direction, we would like to express the sum in (22) as a sum of partitions $\pi = (J_1, \ldots, J_n)$ of $[0,t]$ into T-intervals and sums over $\mathbf{k}_i \in \mathbf{N}^{J_i}$. However, a moment's thought reveals that the sum over $\mathbf{k}$ creates correlations between the different $\mathbf{k}_i$. E.g. J_i being small is a very nonlocal condition in terms of $\mathbf{k}$: nowhere in the whole interval $[0,t]$ can there be a k_n large enough to create a $L \in \mathcal{L}$ that intersects J_i. Given an arbitrary T-interval J and $\mathbf{k} \in \mathbf{N}^J$, we may define, in the same way as we did above for $[0,t]$, the partition $\pi(\mathbf{k})$ of J into small and large intervals. In particular, $\pi(\mathbf{k}) = \{J\}$ means, if $|J| = T$, that $\mathbf{k}$ is such that J is small or large depending on the label on J and, if $|J| > T$, that $\mathbf{k}$ is such that J is large.

Consider now the sum (22). Let $\pi(\mathbf{k}) = \{J_1, \ldots, J_N\}$. Define the Girsanov factor

$$g_{J_i}(\omega) = e^{\int_{J_i} (f(t), \gamma^{-1}(ds(t) - \frac{1}{2} f(t) dt))} \tag{31}$$

where we recall that $f(t)$ and ω, given by (15) and (16), and thus g_{J_i}, depend on the whole past i.e. on $\mathbf{s}([0,\tau_i])$ and $l(0)$. Let $\mathbf{k}_i \in \mathbf{N}^{J_i}$ be the restriction of $\mathbf{k}$ to J_i, let us denote by $\chi_{\mathbf{k}_i}$ the corresponding product (21), and let

$$\mu_{\mathbf{k}_i}(d\mathbf{s}(J_i)) = \chi_{\mathbf{k}_i} g_{J_i} \nu^{|J_i|}_{s(\tau_{i-1})}(d\mathbf{s}(J_i)). \tag{32}$$

We can then write

$$\chi_{\mathbf{k}} \mu^t_{\omega(0)}(d\mathbf{s}) = \prod_{i=1}^{N} \mu_{\mathbf{k}_i}(d\mathbf{s}(J_i)). \tag{33}$$

Let π be a partition of $[0,t]$ into T-intervals with labels "small" or "large" on the ones of length T. Let us define, for such a labelled T-interval J, $1_J(\mathbf{k})$ to be the indicator function for the set of $\mathbf{k} \in \mathbf{N}^J$ such that $\pi(\mathbf{k}) = \{J\}$ (i.e. if $|J| = T$ 1_J is supported on $\mathbf{k}$ so that J is small or large depending on the label and if $|J| > T$ on $\mathbf{k}$ so that J is large). For two adjacent T-intervals J, J' let $1_{JJ'}(\mathbf{k}, \mathbf{k}')$ be the indicator function for the set of $(\mathbf{k}, \mathbf{k}') \in \mathbf{N}^J \times \mathbf{N}^{J'}$, such that $\gamma_L \leq \beta(L)$ for all $L \subset J \cup J'$ which intersect both J and J'. Using Lemma 4.2, we may then write eq. (22) as

$$\mu^t_{\omega(0)}(d\mathbf{s}) = \sum_{\pi} \sum_{\mathbf{k}_1 \ldots \mathbf{k}_N} \prod_{i=1}^{N} 1_{J_i}(\mathbf{k}_i) \mu_{\mathbf{k}_i}(d\mathbf{s}(J_i)) \prod_{i=1}^{N-1} 1_{J_i J_{i+1}}(\mathbf{k}_i, \mathbf{k}_{i+1}). \tag{34}$$

Note that this expression has a Markovian structure in the sets J_i, but each $\mu_{\mathbf{k}_i}$ depends on the whole past history. In the next Section, we shall decouple this dependence.

5 Decoupling

By decoupling we mean that we shall write $\mu^t_{\omega(0)}$ as a product of measures whose dependence on the past extends only over two adjacent intervals, and corrections. To achieve that, consider $\mu_{\mathbf{k}_i}$, for $i > 2$; remember that $[0, t]$ is partitioned into intervals $J_i = [\tau_{i-1}, \tau_i]$ with $\tau_0 = 0$, $\tau_N = t$. Fix $j < i$, and introduce the drift with memory on $[\tau_{j-1}, t]$:

$$f_j(t) = PF(\omega_j(t)) \tag{35}$$

where

$$\omega_j(t) = (s(t), l(t, \mathbf{s}([\tau_{j-1}, t]), 0))$$

is the solution of (9, 10), with initial condition $l(\tau_{j-1}) = 0$. We denote by g_{ij} the Girsanov factor $g_{J_i}(\omega_j)$ (given by (31), with $f(t)$ replaced by $f_j(t)$). Note that it depends only on the history $\mathbf{s}([\tau_{j-1}, \tau_i])$.

Since the characteristic function $\chi_{\mathbf{k}_i}$ also depends on the past through the ω dependence of (21), we need to decouple this too. We let

$$\chi_{\mathbf{k}_i j} = \prod_{\mathbf{n} \subset J_i} \phi_{k_n}(D_n(\omega_j)). \tag{36}$$

We can now define the decoupled measure for $j = 2, \ldots, i-1$:

$$\mu_{\mathbf{k}_i j}(d\mathbf{s}(J_i)|\mathbf{s}([\tau_{j-1}, \tau_{i-1}])) = \chi_{\mathbf{k}_i j} g_{ij} \nu^{|J_i|}_{s(\tau_{i-1})}(d\mathbf{s}(J_i)); \tag{37}$$

this measure is defined on the paths on the time interval J_i and depends on the past up to and including the interval J_j. To connect to (33), we write, for $i \geq 3$, a telescopic sum

$$\mu_{\mathbf{k}_i} = \mu_{\mathbf{k}_i i-1} + \sum_{j=1}^{i-2} (\mu_{\mathbf{k}_i j} - \mu_{\mathbf{k}_i j+1}) \equiv \sum_{j=1}^{i-1} \mu_{\mathbf{k}_i, j} \ , \tag{38}$$

where by definition $\mu_{\mathbf{k}_i 1} = \mu_{\mathbf{k}_i}$; note that this term is the only one depending on $l(0)$. For $i = 1, 2$ we will set by convention $j_i = i - 1$, $\mathbf{s}([\tau_{-1}, \tau_0]) = \omega(0)$, and define $\mu_{\mathbf{k}_i j_i} = \mu_{\mathbf{k}_i}$. Inserting (38) into (34), we get

$$\mu^t_{\omega(0)}(d\mathbf{s}) = \sum_{\pi} \sum_{\mathbf{k}_1 \ldots \mathbf{k}_N} \sum_{\mathbf{j}} \prod_{i=1}^{N} 1_{J_i}(\mathbf{k}_i) \mu_{\mathbf{k}_i, j_i}(d\mathbf{s}(J_i)|\mathbf{s}([\tau_{j_i-1}, \tau_{i-1}])) \prod_{i=1}^{N-1} 1_{J_i J_{i+1}}(\mathbf{k}_i, \mathbf{k}_{i+1}). \tag{39}$$

One should realize that the leading term in the sum (39) is the one with all $j_i = i - 1$ and $\mathbf{k}$ such that the partition $\pi(\mathbf{k})$ consists of only small intervals. Indeed, $\mu_{\mathbf{k}_i, j_i}$ with $j_i \neq i - 1$ describes the change of $\mu_{\mathbf{k}_i}$ under variation in distant past. This will be shown to be small as a consequence of Proposition 1 and Lemma 4.1. On the other hand, the occurrence of large intervals will be shown to have a small probability, using Proposition 2.

The first Proposition controls the unlikely events of having either Δ^n, $n \geq 1$, or μ_K with $|K| \geq 2T$ (or both):

Proposition 3. *There exists $c > 0$, $c' < \infty$, $T_0 = T_0(\rho, R) < \infty$ such that, $\forall T \geq T_0$, and for $|K| \geq 2T$, or $m \geq 2$, or $m = 1$ and $|K| \geq T$,*

$$\sup_{\mathbf{k}'} \sup_{\mathbf{s}' \in C_s} \int |\Delta^m \mu_K(d\sigma|\sigma')| \leq e^{-c(|K| + Tm)} C_K(\omega(0)) \tag{40}$$

where the sup is over $\mathbf{k}'$ so that J_0 is small, if $0 \notin K$. $C_K(\omega(0)) = 1$ if $0 \notin K$ and

$$C_K(\omega(0)) = e^{c'\beta' T} e^{\frac{||\omega(0)||^2}{8R}} \tag{41}$$

if $0 \in K$.

For the likely events we look more closely at $\bar{\mu}^n$:

$$\bar{\mu}^n(d\sigma|\sigma') = \int \bar{\mu}(d\sigma|\sigma'')\lambda^{n-1}(d\mathbf{s}''|\mathbf{s}') \tag{42}$$

with λ given by, see (**??**),

$$\lambda(d\mathbf{s}|\mathbf{s}') = \sum_{\mathbf{k}} \bar{\mu}(d\sigma|\sigma') = \sum_{\mathbf{k}} \chi_{\mathbf{k}}(\mathbf{s}, \mathbf{s}')P(d\mathbf{s}|\mathbf{s}'). \tag{43}$$

The content of the following proposition is that λ^n relaxes to equilibrium:

Proposition 4. *There exist* $\delta = \delta(\rho, R) > 0$, $p = p(\rho, R) < \infty$, *such that,* $\forall T \geq T_0$,

$$\sup_{\mathbf{s}' \in C_s} \int |\lambda^p(d\mathbf{s}|\mathbf{s}') - \lambda^p(d\mathbf{s}|0)| \leq 1 - \delta \tag{44}$$

6 Proof of Proposition 3

For $i \in \mathcal{I}$, we rewrite $\mu_{\mathbf{k}_i, j_i}$ (see eq. (38)) as

$$\mu_{\mathbf{k}_i,j_i} = \chi_{\mathbf{k}_i j_i} g_{ij_i} - \chi_{\mathbf{k}_i j_i+1} g_{ij_i+1} = (\delta_i \chi + \chi_{\mathbf{k}_i j_i+1} \delta_i g) g_{ij_i}$$

where

$$\delta_i \chi = \chi_{\mathbf{k}_i j_i} - \chi_{\mathbf{k}_i j_i+1} \tag{45}$$

and

$$\delta_i g = 1 - \frac{g_{ij_i+1}}{g_{ij_i}}. \tag{46}$$

Introducing the probability measures

$$\mu_{\mathbf{j}} = \prod_{i=1}^{n} g_{ij_i} \nu_{s(\tau)}^{|K|}, \tag{47}$$

where $j_i \equiv i - 1$ if $i > n - m$, we can write (**??**) as

$$X(\sigma') \leq \sum_{\pi\mathbf{kj}} \sum_{A \subset \mathcal{I}} \int \prod_{i \in A} |\delta_i \chi| \prod_{i \in B} \chi_{\mathbf{k}_i j_i+1} |\delta_i g| \prod_{i \in C} \chi_{\mathbf{k}_i i-1} \mu_{\mathbf{j}} 1(\mathbf{k}|\mathbf{k}') \equiv \sum_{\pi\mathbf{kj}A} \int \mathcal{R}_{\mathbf{kj}A} \mu_{\mathbf{j}} 1(\mathbf{k}|\mathbf{k}') \tag{48}$$

where $B = \mathcal{I} \setminus A$ and $C = \{1, \ldots, n\} \setminus \mathcal{I}$.

Letting

$$\delta_i f = f_{j_i+1} - f_{j_i},$$

(46) can be written as

$$\delta_i g = 1 - e^{\int_{J_i} (\delta_i f(t), \gamma^{-1}(ds(t) - f_{j_i}(t)dt)) - \frac{1}{2}\int_{J_i} (\delta_i f(t), \gamma^{-1} \delta_i f(t))dt} \equiv 1 - H_i \tag{49}$$

and

$$\delta_i \chi = \prod_{\mathbf{n} \subset J_i} \phi_{k_n}(D_n(\omega_{j_i})) - \prod_{\mathbf{n} \subset J_i} \phi_{k_n}(D_n(\omega_{j_i+1})) \tag{50}$$

We will now undo the Girsanov transformation, i.e. change variables from s back to b. Let E denote the expectation with respect to the Brownian motion b with covariance γ on the time interval K. Then,

$$\int \mathcal{R}_{\mathbf{kj}A}\mu_{\mathbf{j}} = E\mathcal{R}_{\mathbf{kj}A} \tag{51}$$

where $\mathcal{R}$ is given by the same expression as before, but the symbols s and ω_{j_i} have to be interpreted as follows: s is the progressively measurable function of b defined on each interval J_i as the solution of

$$ds(t) = f_{j_i}(t)dt + db(t), \tag{52}$$

where $f_j(t) = PF(\omega_j(t))$ and $\omega_j(t) = s(t) + l(t, \mathbf{s}([\tau_{j-1}, t]), 0)$, with, for $i = 1$, $\mathbf{s}([\tau_{j_1-1}, \tau_0])$ replaced by $\mathbf{s}'(J_0)$, which expresses the dependence of (51) on $\mathbf{s}'$; H_i, defined by (49), can be written:

$$H_i = e^{\int_{J_i}(\delta_i f(t), \gamma^{-1}db(t)) - \frac{1}{2}\int_{J_i}(\delta_i f(t), \gamma^{-1}\delta_i f(t))dt.} \tag{53}$$

We will call the $\omega_{j_i}(t)$ collectively by

$$\omega_{\mathbf{j}}(t) = \omega_{j_i}(t) \ \text{ for } \ t \in J_i, \tag{54}$$

and reserve the notation $\omega(t)$ for the solution of the Navier Stokes equation (4) with given $b(\bar{K})$ and with initial condition $\omega(\tau_0) = (s(\tau_0), l(\tau_0, \mathbf{s}'(J_0), 0))$ determined by the $\mathbf{s}'$ in (**??**).

Remark. $\omega_{\mathbf{j}}(t)$ is *not* a solution of (4) on the interval $\bar{K}$ with initial condition given at time τ. On each interval J_i it solves (4) but when moving to the next interval the l-part is possibly set equal to zero, depending on $\mathbf{j}$.

The following Proposition contains the key bounds needed to estimate (51).

Proposition 5. *Let b belong to the support of $\mathcal{R}_{\mathbf{kj}A}$ in (51). Then, there exists a constant c such that*

$$||\delta_i f(t)|| \leq e^{-c\kappa R \mathrm{dist}(J_i, J_{j_i})} \equiv \epsilon_i \tag{55}$$

For an upper bound for the expectation of $\bar{H}_1^2$, we use

Lemma 7.1. *Let $\zeta(t) \in C([0,t], H_s)$ be progressively measurable. Then*

$$Ee^{\int_0^t (\zeta, \gamma^{-1}db) + \lambda\int_0^t(\zeta, \gamma^{-1}\zeta)dt} \leq e^{2(1+\lambda)t||\zeta||^2\rho^{-1}} \tag{56}$$

where $||\zeta|| = \sup_\tau ||\zeta(\tau)||_2$.

Proof. This is just a Novikov bound: we bound the LHS, using Schwarz' inequality, by

$$(Ee^{\int_0^t(2\zeta, \gamma^{-1}db) - 2\int_0^t(\zeta, \gamma^{-1}\zeta dt)})^{\frac{1}{2}}(Ee^{2(1+\lambda)\int_0^t(\zeta, \gamma^{-1}\zeta)dt})^{\frac{1}{2}}$$

and note that the expression inside the first square root is the expectation of a martingale and equals one. □

The bound (55) follows from (57) below, using (12) to bound $\|\delta l\|$, $\sup_{t\in\mathbf{n}}\|\omega(t)\| \leq (2D_n)^{\frac{1}{2}}$, and Lemmas 7.4, 7.5 to bound the exponent in (12).

Lemma 7.6. *Let $f(\omega) = PF(\omega)$ and $\omega = s + l$, $\omega' = s + l'$. Then,*

$$\|f(\omega) - f(\omega')\| \leq C(R)(2\|\omega\|\|\delta l\| + \|\delta l\|^2) \tag{57}$$

with $\delta l = l - l'$. *Moreover*

$$\|f(\omega)\| \leq C(R)(\|\omega\| + \|\omega\|^2) \tag{58}$$

Proof. We have

$$|f_k(\omega) - f_k(\omega')| \leq \sum_p |\omega_{k-p}\omega_p - \omega'_{\kappa-p}\omega'_p| \frac{|k|}{|p|}$$

which, since $|k| \leq \sqrt{\kappa R}$ is bounded by

$$\sqrt{\kappa R} \sum_p |s_{k-p}\delta l_p + s_p \delta l_{k-p} + l_p l_{k-p} - l'_p l'_{k-p}|. \tag{59}$$

Writing $l_p l_{k-p} - l'_p l'_{k-p} = l_p \delta l_{k-p} + l_{k-p}\delta l_p - \delta l_p \delta l_{k-p}$ and using Schwarz' inequality, we get

$$(59) \leq \sqrt{\kappa R}(2\|\omega\|\|\delta l\|\| + \|\delta l\|^2)$$

which proves (57), since $f_k \neq 0$ only for $k \leq \kappa R$. The proof of (58) is similar. □

7 Markov chain estimates.

The goal of this Section is to prove Proposition 4. Although $\lambda(d\mathbf{s}|\mathbf{s}')$ defined in (43) does not define a Markov chain, because of the indicator function $\sum_{\mathbf{k}} \chi_{\mathbf{k}}(\mathbf{s}, \mathbf{s}')$, it is close to one, at least up to the time p in which we are interested, and the proof will be based essentially on Markov chain ideas. To see how close λ is to a Markov chain, compare it with $P(d\mathbf{s}|\mathbf{s}') = g_J(\omega)\nu^T_{s(\tau)}(d\mathbf{s})$ (see (**??**)), which is thus like λ, but without the $\sum_{\mathbf{k}} \chi_{\mathbf{k}}(\mathbf{s}, \mathbf{s}')$; the function $1 - \sum_{\mathbf{k}} \chi_{\mathbf{k}}(\mathbf{s}, \mathbf{s}')$ is supported on $\mathbf{k}$'s such that J is a large interval. For $\mathbf{s}' \in C_s$, we have $\|\omega(\tau)\|^2 \leq 2\beta' TR$ and we can use Proposition 2 to show that there exists a $c > 0$, such that, $\forall \mathbf{s}' \in C_s$, $\forall B \subset C_s$ (note that the support of λ is included in C_s),

$$|\lambda(B|\mathbf{s}') - P(B|\mathbf{s}')| \leq e^{-cT}, \tag{60}$$

and

$$P(C_s|\mathbf{s}') \geq 1 - e^{-cT}; \tag{61}$$

Indeed, if J is large, either there is an interval $L \subset J$ where (29) holds, and we use (23) for that interval, with 0 replaced by τ, $\|\omega(\tau)\|^2 \leq 2\beta' TR$ and $\beta \geq C\beta'$. Or (30) holds, i.e. $D_{\tau+T} \geq \beta' TR$, and we can use (23) with 0 replaced by τ and $t = t' - 1$ replaced by $\tau + T$. Now, we state the main result of this Section:

Proposition 6. *There exists a constant* $\delta > 0$, $\delta = \delta(R, \rho)$ *but independent of* T, *such that* $\forall \mathbf{s_1}, \mathbf{s_2} \in C_s$ *and* $\forall B \subset C_s$,

$$\lambda^2(B|\mathbf{s_1}) + \lambda^2(B^c|\mathbf{s_2}) \geq \delta \tag{62}$$

Remark. The important point in this Proposition is that δ is independent of T. The same will be true about the constants δ_1, δ_2, used in the proof (see (**??**), (**??**)).

Before proving this Proposition, we use it to give the

Proof of Proposition 4. We shall use the previous Proposition and a slightly modified version of an argument taken from [3], p. 197–198. Let, for $B \subset C_s$,

$$\underline{\lambda}(n, B) = \inf_{\mathbf{s} \in C_s} \lambda^n(B|\mathbf{s}), \quad \overline{\lambda}(n, B) = \sup_{\mathbf{s} \in C_s} \lambda^n(B|\mathbf{s}).$$

Fix $\mathbf{s_1}, \mathbf{s_2} \in C_s$ and consider the function defined on subsets $B \subset C_s$:

$$\psi_{\mathbf{s_1},\mathbf{s_2}}(B) = \lambda^2(B|\mathbf{s_1}) - \lambda^2(B|\mathbf{s_2}).$$

Let S^+ be the set such that $\psi_{\mathbf{s_1},\mathbf{s_2}}(B) \geq 0$ for $B \subset S^+$ and $\psi_{\mathbf{s_1},\mathbf{s_2}}(B) \leq 0$ for $B \subset C_s \backslash S^+ \equiv S^-$ ($S^\pm$ depend on $\mathbf{s_1}, \mathbf{s_2}$, but we suppress this dependence). Observe that, by (60, 61), we have, $\forall \mathbf{s} \in C_s$,

$$1 - e^{-cT} \leq \lambda^2(C_s|\mathbf{s}) \leq 1. \tag{63}$$

(with a smaller c than in (60, 61)). Then,

$$|\psi_{\mathbf{s_1},\mathbf{s_2}}(S^+) + \psi_{\mathbf{s_1},\mathbf{s_2}}(S^-)| = |\lambda^2(C_s|\mathbf{s_1}) - \lambda^2(C_s|\mathbf{s_2})| \leq e^{-cT}. \tag{64}$$

Moreover, using (63, 62), and $S^+ \cup S^- = C_s$,

$$\begin{aligned} \psi_{\mathbf{s_1},\mathbf{s_2}}(S^+) &= \lambda^2(S^+|\mathbf{s_1}) - \lambda^2(S^+|\mathbf{s_2}) \\ &\leq 1 - (\lambda^2(S^-|\mathbf{s_1}) + \lambda^2(S^+|\mathbf{s_2})) \leq 1 - \delta. \end{aligned} \tag{65}$$

Thus,

$$\begin{aligned} \overline{\lambda}(t+2,B) - \underline{\lambda}(t+2,B) &= \sup_{\mathbf{s_1},\mathbf{s_2}} \int (\lambda^2(d\mathbf{s}|\mathbf{s_1}) - \lambda^2(d\mathbf{s}|\mathbf{s_2}))\lambda^t(B|\mathbf{s}) \\ &= \sup_{\mathbf{s_1},\mathbf{s_2}} \int \psi_{\mathbf{s_1},\mathbf{s_2}}(d\mathbf{s})\lambda^t(B|\mathbf{s}) \\ &\leq \sup_{\mathbf{s_1},\mathbf{s_2}} (\psi_{\mathbf{s_1},\mathbf{s_2}}(S^+)\overline{\lambda}(t,B) + \psi_{\mathbf{s_1},\mathbf{s_2}}(S^-)\underline{\lambda}(t,B)) \\ &= \sup_{\mathbf{s_1},\mathbf{s_2}} (\psi_{\mathbf{s_1},\mathbf{s_2}}(S^+)(\overline{\lambda}(t,B) - \underline{\lambda}(t,B)) + (\psi_{\mathbf{s_1},\mathbf{s_2}}(S^+) + \psi_{\mathbf{s_1},\mathbf{s_2}}(S^-))\underline{\lambda}(t,B)) \\ &\leq (1-\delta)(\overline{\lambda}(t,B) - \underline{\lambda}(t,B)) + e^{-cT} \end{aligned}$$

where, to get the last inequality, we used (65) and (64) and $\underline{\lambda}(t,B) \leq 1$. We conclude that, $\forall \mathbf{s}' \in C_s$,

$$|\lambda^{2n}(B|\mathbf{s}') - \lambda^{2n}(B|0)| \leq \overline{\lambda}(2n,B) - \underline{\lambda}(2n,B) \leq (1-\delta)^{n-1} + \frac{e^{-cT}}{\delta}.$$

Now, choose first n sufficiently large so that $(1-\delta)^{n-1} \leq \frac{1-\bar{\delta}}{4}$, for some $\bar{\delta} > 0$ and then T sufficiently large so that $\frac{e^{-cT}}{\delta} \leq \frac{1-\bar{\delta}}{4}$. Since, with $p = 2n$,

$$\int |\lambda^p(d\mathbf{s}|\mathbf{s}') - \lambda^p(d\mathbf{s}|0)| \leq 2 \sup_B |\lambda^{2n}(B|\mathbf{s}') - \lambda^{2n}(B|0)|,$$

(44) follows, with δ in that equation equal to $\bar{\delta}$ here. □

Proof of Proposition 6. First of all, observe that it is enough to prove (62) with λ replaced by P:

$$P^2(B|\mathbf{s_1}) + P^2(B^c|\mathbf{s_2}) \geq \delta \tag{66}$$

since we can then use (60) and choose T large enough to obtain the same result for λ, since δ is independent of T.

It will be convenient to write $P^2(d\mathbf{s}_+|\mathbf{s_1}) = \int P(d\mathbf{s}_+|\mathbf{s})P(d\mathbf{s}|\mathbf{s_1})$, where we write $\mathbf{s}_+ \in C_s^+$ meaning $\mathbf{s}_+ \in C_s \subset C([0,T], H_s)$ (see (**??**)), and similarly $\mathbf{s_1} \in C_s^- \subset C([-2T,-T], H_s)$, $\mathbf{s} \in C_s^0 \subset C([-T,0], H_s)$, which is the variable over which we integrate.

Turning to the proof, we first get a lower bound on (66) by replacing B, B^c by $B \cap V^+$, $B^c \cap V^+$, where V^+ is defined by

$$V^+ = \{\mathbf{s}_+ \in C_s^+ | \sum_{n=1}^{t} D_n(\omega_0) < \zeta R t, \;\; \forall t \in [1,T]\} \tag{67}$$

where ζ will be chosen large enough below and $\omega_0(t) = s(t) + l(t, \mathbf{s}_+([0,t]), 0)$. To simplify the notation, we shall assume, from now on, that $B \subset V^+$ and $B^c \equiv V^+ \backslash B$.

Next, we obtain also a lower bound on $P^2(B|\mathbf{s_1}) = \int P(B|\mathbf{s})P(d\mathbf{s}|\mathbf{s_1})$ and on $P^2(B^c|\mathbf{s_2})$ by restricting the integrations over $\mathbf{s}$, so that we have:

$$(66) \geq \int P(B|\mathbf{s})1(\mathbf{s}|\mathbf{s_1})P(d\mathbf{s}|\mathbf{s_1}) + \int P(B^c|\mathbf{s})1(\mathbf{s}|\mathbf{s_2})P(d\mathbf{s}|\mathbf{s_2}) \tag{68}$$

where $1(\mathbf{s}|\mathbf{s}') = 1_0 1_{[-1,0]} 1_{\leq -1}$ with

$$\begin{aligned} &1_0(s(0)) = 1(\|s(0)\|^2 \leq 3\zeta' R), \\ &1_{[-1,0]}(\mathbf{s}([-1,0])) = 1(\sup_{t\in[-1,0]} \|s(t)\|^2 \leq \zeta R), \\ &1_{\leq -1}(\mathbf{s}([-T,-1])|\mathbf{s}') = 1(\|\omega(-1)\|^2 \leq \zeta' R), \end{aligned} \tag{69}$$

where $\omega(-1) = s(-1) + l(-1, \mathbf{s}([-T,-1]), l(-T))$, with $l(-T) = l(-T, \mathbf{s}'([-2T,-T]), 0)$, and ζ, ζ' are constants that will be chosen large enough below, but with $\zeta' \leq C\zeta$ for C large (ζ, ζ' play a role somewhat similar to β, β' in the previous Sections, but they are not necessarily equal to the latter).

Before proceeding further, let us explain the basic idea of the proof. To prove (66), it would be enough to bound $\frac{P^2(B|\mathbf{s_1})}{P^2(B|\mathbf{s_2})} \geq \delta$. We do not quite do that, but first give, in Lemma 8.1 below, a lower bound on $\frac{P(B|\mathbf{s})}{P(B|\mathbf{s}')}$ for $\mathbf{s}, \mathbf{s}'$ in a "good" set of configurations, i.e. in the support of the indicator functions that we just introduced. Good here means that the "interaction" (or, to be more precise, the analogue of what is called in Statistical Mechanics the relative Hamiltonian), expressed through the Girsanov formula (see e.g. (**??**)), between the paths in C^0_s that are in the support of those indicator functions and those in V^+ is, in some sense, bounded. This relies on Lemma 8.5, which itself follows from the results of the previous Section. Next, we show that the probability of reaching that good set, does not depend very much on whether we start from $\mathbf{s_1}$ or $\mathbf{s_2}$ in C^-_s (see Lemma 8.2). This is rather straightforward, but depends on standard estimates on the Brownian bridge (see Lemmas 8.6 and 8.7) that we give in detail, for the sake of completeness. Finally, we need to show that the probability of the set of good configurations, as well as the one of V^+, is bounded from below; this is done in Lemma 8.3. Remember that all the bounds here have to be T-independent, since this was used in an essential way in the proof of the Theorem (Section 6).

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