Partial *-algebras, twenty years later

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Abstract

We review the main points in the development of partial *-algebras during the last 20 years, at three different levels: (i) The algebraic structure stemming from the partial multiplication; (ii) The locally convex partial *-algebras; (iii) The partial *-algebras of closable operators in Hilbert spaces or partial O*-algebras, including the representation theory of the abstract partial *-algebras.

1 Prologue

Twenty years ago, Witold Karwowski suggested to the author to look at the algebraic structure, if any, that would arise if one tried to multiply unbounded operators in a Hilbert space. Indeed there was a rich structure behind, and not only at the algebraic level [4, 5]. Since then, several researchers have joined this circle of ideas, and a full-fledged theory has emerged, thanks to the work of F. Mathot, J. Shabani (Louvain-la-Neuve), W. Karwowski (Wrocław), G. and G.A. Lassner (Leipzig), G. Epifanio, C. Trapani, F. Bagarello, F. Tschinke (Palermo), A. Inoue, H. Ogi, I. Ikeda, M. Takakura (Fukuoka). We will present here a quick overview of this rather unforeseen development, following essentially [9], [10], and [12], where the proofs and the original references may be found.

2 The algebraic structure

A *partial *-algebra* is a complex vector space $\mathfrak{A}$, endowed with an involution $x \mapsto x^*$ (that is, a bijection such that $x^{**} = x$) and a partial multiplication defined by a set $\Gamma \subset \mathfrak{A} \times \mathfrak{A}$ (a binary relation) such that:

(i) $(x, y) \in \Gamma$ implies $(y^*, x^*) \in \Gamma$;

(ii) $(x, y_1), (x, y_2) \in \Gamma$ implies $(x, \lambda y_1 + \mu y_2) \in \Gamma$, $\forall\, \lambda, \mu \in \mathbb{C}$;

(iii) for any $(x, y) \in \Gamma$, there is defined a product $xy \in \mathfrak{A}$, which is distributive w.r. to the addition and satisfies the relation $(xy)^* = y^* x^*$.

Notice that the partial multiplication is *not* required to be associative (and often it is not). We shall assume the partial *-algebra $\mathfrak{A}$ contains a unit e, i.e., $e^* = e$, $(e, x) \in \Gamma$, $\forall\, x \in \mathfrak{A}$, and $ex = xe = x$, $\forall\, x \in \mathfrak{A}$. (If $\mathfrak{A}$ has no unit, it may always be embedded into a larger partial *-algebra with unit, in the standard fashion [6].)

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Given the defining set Γ, spaces of multipliers are defined in the obvious way:

$$\begin{aligned}(x,y)\in\Gamma \quad &\Leftrightarrow \quad x\in L(y) \ \text{ or } x \text{ is a left multiplier of } y\\ &\Leftrightarrow \quad y\in R(x) \ \text{ or } y \text{ is a right multiplier of } x.\end{aligned}$$

For any subset $\mathfrak{N}\subset\mathfrak{A}$, we write

$$L\mathfrak{N}=\bigcap_{x\in\mathfrak{N}} L(x), \quad R\mathfrak{N}=\bigcap_{x\in\mathfrak{N}} R(x),$$

and, of course, the involution exchanges the two:

$$(L\mathfrak{N})^* = R\mathfrak{N}^*, \quad (R\mathfrak{N})^* = L\mathfrak{N}^*.$$

Clearly all these multiplier spaces are vector subspaces of $\mathfrak{A}$, containing e.

The partial *-algebra is *abelian* if $L(x) = R(x), \forall\, x\in\mathfrak{A}$, and then $xy = yx, \forall\, y\in L(x)$. In that case, we write simply for the multiplier spaces $L(x) = R(x) \equiv M(x)$, $L\mathfrak{N} = R\mathfrak{N} \equiv M\mathfrak{N}\ (\mathfrak{N}\subset\mathfrak{A})$.

Now the crucial fact is that the couple of maps (L,R) defines a *Galois connection* [3] on the complete lattice of all vector subspaces of $\mathfrak{A}$ (ordered by inclusion), which means that (i) both L and R reverse order; and (ii) both LR and RL are closures, that is:

$$\mathfrak{N}\subset LR\mathfrak{N} \ \text{ and } \ LRL = L, \qquad \mathfrak{N}\subset RL\mathfrak{N} \ \text{ and } \ RLR = R.$$

Let us denote by $\mathcal{F}^L$, resp. $\mathcal{F}^R$, the set of all LR-closed, resp. RL-closed, subspaces of $\mathfrak{A}$:

$$\mathcal{F}^L = \{\mathfrak{N}\subset\mathfrak{A} : \mathfrak{N} = LR\mathfrak{N}\}, \quad \mathcal{F}^R = \{\mathfrak{N}\subset\mathfrak{A} : \mathfrak{N} = RL\mathfrak{N}\},$$

both ordered by inclusion. Then, from standard results of universal algebra, one can deduce the following result.

Theorem 2.1 *(1) The set $\mathcal{F}^R$, ordered by inclusion, is a complete lattice with lattice operations*

$$\mathfrak{M}\wedge\mathfrak{N} = \mathfrak{M}\cap\mathfrak{N}, \quad \mathfrak{M}\vee\mathfrak{N} = RL(\mathfrak{M}+\mathfrak{N}).$$

The largest element is $\mathfrak{A}$, the smallest $R\mathfrak{A}$. A corresponding result holds for $\mathcal{F}^L$, exchanging L and R.

(2) Both $L:\mathcal{F}^R\to\mathcal{F}^L$ and $R:\mathcal{F}^L\to\mathcal{F}^R$ are lattice anti-isomorphisms: $L(\mathfrak{M}\wedge\mathfrak{N}) = L\mathfrak{M}\vee L\mathfrak{N}$, etc.

(3) The involution $\mathfrak{N}\leftrightarrow\mathfrak{N}^*$ is a lattice isomorphism between $\mathcal{F}^L$ and $\mathcal{F}^R$.

As examples of partial *-algebras, some of which we will encounter below, we may cite partial *-algebras of polynomials, of functions, of infinite matrices or kernels, topological quasi *-algebras, Banach partial *-algebras, CQ*-algebras, and partial *-algebras of closable operators in a Hilbert space (partial O*-algebras).

The last case is the most important in practice. It will also be needed to set up a representation theory, because a representation of a partial *-algebra $\mathfrak{A}$ is a homomorphism of $\mathfrak{A}$ into some partial O*-algebra. Here a **-homomorphism* of a partial *-algebra $\mathfrak{A}$ into another one $\mathfrak{B}$ is a linear map $\rho : \mathfrak{A} \to \mathfrak{B}$ such that (i) $\rho(x^*) = \rho(x)^*$ for every $x \in \mathfrak{A}$, and (ii) whenever $x \in L(y)$ in $\mathfrak{A}$, then $\rho(x) \in L(\rho(y))$ in $\mathfrak{B}$ and $\rho(x)\rho(y) = \rho(xy)$. The map ρ is a **-isomorphism* if it is a bijection and $\rho^{-1} : \mathfrak{B} \to \mathfrak{A}$ is also a *-homomorphism. In particular, for $\mathfrak{A} = \mathfrak{B}$, one speaks of a **-automorphism.*

3 Locally convex partial *-algebras

3.1 Basic definitions

Let $\mathfrak{A}$ be a partial *-algebra with unit and assume it carries a locally convex, Hausdorff, topology τ, which makes it into a locally convex topological vector space $\mathfrak{A}[\tau]$ (that is, the vector space operations are τ-continuous).

The partial *-algebraic structure of $\mathfrak{A}$ is completely characterized by its spaces of left, resp. right, multipliers. Thus, quite naturally, we describe the topological structure of $\mathfrak{A}[\tau]$ by providing all spaces of multipliers with appropriate topologies. Our goal is to make the algebraic and the topological structure coincide as much as possible.

We start with the following observation. Let $\mathfrak{M} \in \mathcal{F}^R$. To every $a \in L\mathfrak{M}$, one may associate a linear map L_a from $\mathfrak{M}$ into $\mathfrak{A}$:

$$L_a(x) = ax, \quad x \in \mathfrak{M},\ a \in L\mathfrak{M}\,.$$

This allows to define the topology $\rho_{\mathfrak{M}}$ on $\mathfrak{M}$ as the weakest locally convex topology on $\mathfrak{M}$ such that all maps L_a, $a \in L\mathfrak{M}$, are continuous from $\mathfrak{M}$ into $\mathfrak{A}[\tau]$. This is of course a projective topology. In the same way, the topology $\lambda_{\mathfrak{N}}$ on $\mathfrak{N} \in \mathcal{F}^L$ is the weakest locally convex topology on $\mathfrak{N}$ such that all maps $R_b : x \mapsto xb$, $x \in \mathfrak{N}$, $b \in R\mathfrak{N}$, are continuous from $\mathfrak{N}$ into $\mathfrak{A}[\tau]$.

It follows immediately from the definition that, whenever $\mathfrak{M}_1, \mathfrak{M}_2 \in \mathcal{F}^R$ are such that $\mathfrak{M}_1 \subset \mathfrak{M}_2$, then the topology $\rho_{\mathfrak{M}_1}$ is finer than the topology $(\rho_{\mathfrak{M}_2} \restriction \mathfrak{M}_1)$ induced by $\mathfrak{M}_2$ on $\mathfrak{M}_1$. In other words, the embedding $\mathfrak{M}_1 \to \mathfrak{M}_2$ is a *continuous injection.*

Take now $\mathfrak{A}$ itself. It carries three topologies, τ, $\rho_{\mathfrak{A}}$ and $\lambda_{\mathfrak{A}}$, and it is easy to see that both $\rho_{\mathfrak{A}}$ and $\lambda_{\mathfrak{A}}$ are finer than τ. As a consequence, since τ was assumed to be Hausdorff, all topologies $\rho_{\mathfrak{M}}$, $\mathfrak{M} \in \mathcal{F}^R$, and $\lambda_{\mathfrak{N}}$, $\mathfrak{N} \in \mathcal{F}^L$, are Hausdorff.

Now, for reasons of coherence, it would be preferable that all three topologies on $\mathfrak{A}$, τ, $\rho_{\mathfrak{A}}$ and $\lambda_{\mathfrak{A}}$ be equivalent. Here is a handy criterion.

Lemma 3.1 *Let $\mathfrak{A}[\tau]$ be a partial *-algebra with locally convex topology τ. Then the projective topology $\rho_{\mathfrak{A}}$ on $\mathfrak{A}$ is equivalent to τ iff, for each $a \in L\mathfrak{A}$, the map $L_a : x \mapsto ax$ is continuous from $\mathfrak{A}[\tau]$ into itself. Similarly, the projective topology $\lambda_{\mathfrak{A}}$ on $\mathfrak{A}$ is equivalent to τ iff, for each $b \in R\mathfrak{A}$, the map $R_b : x \mapsto xb$ is continuous from $\mathfrak{A}[\tau]$ into itself.*

Moreover, if the involution $x \mapsto x^*$ *is* τ*-continuous, then it is continuous from* $\mathfrak{M}[\rho_{\mathfrak{M}}]$ *into* $\mathfrak{M}^*[\lambda_{\mathfrak{M}^*}] \in \mathcal{F}^L$*, for every* $\mathfrak{M} \in \mathcal{F}^R$.

According to our goal, we will naturally require that all three topologies $\rho_{\mathfrak{A}}$, $\lambda_{\mathfrak{A}}$ and τ on a topological partial *-algebra coincide and that the involution be continuous. Let us now look at multiplier spaces $\mathfrak{M} \in \mathcal{F}^R$. If $\mathfrak{M}_1 \subset \mathfrak{M}_2$, we have seen that the embedding is continuous. In order to make the structure tighter, we should also require that $\mathfrak{M}_1$ be *dense* in $\mathfrak{M}_2[\rho_{\mathfrak{M}_2}]$. This is true in many examples, typically the function spaces. Of course it is enough to require that $R\mathfrak{A}$ be dense in each $\mathfrak{M}[\rho_{\mathfrak{M}}] \in \mathcal{F}^R$. Indeed, if $R\mathfrak{A} \subset \mathfrak{M}_1 \subset \mathfrak{M}_2$, and $R\mathfrak{A}$ is dense in $\mathfrak{M}_2$ for $\rho_{\mathfrak{M}_2}$, so is *a fortiori* $\mathfrak{M}_1$. But this condition is still too strong (and hardly verifiable in practice, because $\mathcal{F}^R$ is too large). Thus, we introduce the notion of *generating family*, that is, a subset $\mathcal{I}^R$ of $\mathcal{F}^R$ such that

(i) $R\mathfrak{A} \in \mathcal{I}^R$ and $\mathfrak{A} \in \mathcal{I}^R$, and

(ii) $x \in L(y)$ iff $\exists\, \mathfrak{M} \in \mathcal{I}^R$ such that $y \in \mathfrak{M}, x \in L\mathfrak{M}$.

A generating family for $\mathcal{F}^L$ is defined in a similar way. Clearly, if $\mathcal{I}^R$ is a generating family for $\mathcal{F}^R$, $\mathcal{I}^L = L\mathcal{I}^R = \{L\mathfrak{M} : \mathfrak{M} \in \mathcal{I}^R\}$ is generating for $\mathcal{F}^L$. The usefulness of this notion is twofold:

(i) if $\mathcal{I}^R$ is generating for $\mathcal{F}^R$, so is the sublattice $\mathcal{J}^R$ of $\mathcal{F}^R$ generated from $\mathcal{I}^R$ by *finite* lattice operations $\vee$ and $\wedge$;

(ii) if $\mathcal{I}^R$ is generating, the *complete* lattice generated by $\mathcal{I}^R$ is $\mathcal{F}^R$ itself. In other words, a generating family determines completely the partial multiplication.

We make immediate use of this last property for weakening the density condition.

Proposition 3.2 – *Let* $\mathfrak{A}[\tau]$ *be a partial *-algebra with topology* τ*. Assume there exists a generating family* $\mathcal{I}^R$ *for* $\mathcal{F}^R$ *such that* $R\mathfrak{A}$ *is dense in* $\mathfrak{M}[\rho_{\mathfrak{M}}]$ *for every* $\mathfrak{M} \in \mathcal{I}^R$*. Then, for any pair* $\mathfrak{M}_1, \mathfrak{M}_2 \in \mathcal{F}^R$ *such that* $\mathfrak{M}_1 \subset \mathfrak{M}_2$*,* $\mathfrak{M}_1$ *is dense in* $\mathfrak{M}_2[\rho_{\mathfrak{M}_2}]$.

Summarizing, we may now state our definition of locally convex partial *-algebra.

Definition 3.3 – Let $\mathfrak{A}[\tau]$ be a partial *-algebra, which is a topological vector space for the locally convex topology τ. Then $\mathfrak{A}[\tau]$ is called a *locally convex partial *-algebra* if the following two conditions are satisfied:

(i) the involution $x \mapsto x^*$ is τ-continuous;

(ii) the maps $x \mapsto ax$ and $x \mapsto xb$ are τ-continuous for all $a \in L\mathfrak{A}$ and $b \in R\mathfrak{A}$.

The locally convex partial *-algebra $\mathfrak{A}[\tau]$ is said to be *tight*, if, in addition,

(iii) there is a generating family $\mathcal{J}^R$ for $\mathcal{F}^R$ such that $R\mathfrak{A}$ is dense in $\mathfrak{M}[\rho_{\mathfrak{M}}]$, $\forall \mathfrak{M} \in \mathcal{J}^R$.

This definition seems natural, in the sense that it forces the topological structure determined by τ to be consistent with the multiplier structure of $\mathfrak{A}$.

The simplest example is that of a *topological quasi *-algebra*, which is defined as follows. Let $(\mathfrak{A}, \mathfrak{A}_o)$ be a topological quasi-algebra, that is, $\mathfrak{A}_o$ is a topological *-algebra such that the multiplication is separately, but not jointly, continuous, $\mathfrak{A}_o$ is not complete, and $\mathfrak{A}$ is the completion of $\mathfrak{A}_o$. Thus $\mathfrak{A}$ is only a partial *-algebra: the product xy is defined only if either x or y belongs to $\mathfrak{A}_o$. Clearly, $(\mathfrak{A}, \mathfrak{A}_o)$ is a (trivial) partial *-algebra with $L\mathfrak{M} = R\mathfrak{M} = \mathfrak{A}_o$ and $\mathfrak{A}_o$ is dense in $\mathfrak{A}$. Thus every topological quasi *-algebra is a tight locally convex partial *-algebra.

Other examples are given by *Banach partial *-algebras* and by *Banach or locally convex partial *-algebras of functions*, that we will study in the next sections.

3.2 Banach partial *-algebras

Assume now that $\mathfrak{A}$ is a Banach space for a norm $\|\cdot\|$, such that

(b1) $\|x\| = \|x^*\|$, $\forall\, x \in \mathfrak{A}$;

(b2) $\forall\, a \in L\mathfrak{A}$, $\exists$ a constant $\gamma_a > 0$ such that $\|ax\| \leqslant \gamma_a \|x\|$, $\forall\, x \in \mathfrak{A}$.

Then $\mathfrak{A}[\|\cdot\|]$ is a locally convex partial *-algebra and both $\rho_{\mathfrak{A}}$ and $\lambda_{\mathfrak{A}}$ are equivalent to the norm topology on $\mathfrak{A}$.

We will now describe the topologies of the spaces of multipliers. To begin with, let us consider the spaces of universal multipliers.

The following sets of seminorms define, respectively, the topology $\rho_{R\mathfrak{A}}$ on $R\mathfrak{A}$ and $\lambda_{L\mathfrak{A}}$ on $L\mathfrak{A}$:

$$R\mathfrak{A} \ni b \mapsto \|xb\|,\ x \in \mathfrak{A}; \qquad L\mathfrak{A} \ni a \mapsto \|ax\|,\ x \in \mathfrak{A}.$$

Actually, it turns out that these topologies are often equivalent to norm topologies.

Take now an arbitrary matching pair $\mathfrak{M} \in \mathcal{F}^R$, $L\mathfrak{M} \in \mathcal{F}^L$. Apart from $\rho_{\mathfrak{M}}$ and $\lambda_{L\mathfrak{M}}$, other topologies can be defined on $\mathfrak{M}$ and $L\mathfrak{M}$, respectively, starting from the fact that, as above, $L\mathfrak{M}$ can be identified with a space of linear maps from $\mathfrak{M}$ into $\mathfrak{A}$. Let $\mathcal{G}$ be a bounded subset of $\mathfrak{M}[\rho_{\mathfrak{M}}]$ and $a \in L\mathfrak{M}$. We put

$$\|a\|_{\mathcal{G}} = \sup_{x \in \mathcal{G}} \|ax\|.$$

This family of seminorms endows $L\mathfrak{M}$ with a topology $\Lambda_{L\mathfrak{M}}$ finer than $\lambda_{L\mathfrak{M}}$. One defines in a similar way a topology $\mathrm{P}_{\mathfrak{M}}$ on $\mathfrak{M}$. In general these topologies are neither normable, nor Fréchet.

We will say that a norm $\|\cdot\|_{\mathfrak{M}}$ on $\mathfrak{M} \in \mathcal{F}^R$ is *admissible* if one has $\rho_{\mathfrak{M}} \preceq \|\cdot\|_{\mathfrak{M}} \preceq \mathrm{P}_{\mathfrak{M}}$. Similarly, a norm $\|\cdot\|_{L\mathfrak{M}}$ on $L\mathfrak{M} \in \mathcal{F}^L$ is admissible whenever $\lambda_{L\mathfrak{M}} \preceq \|\cdot\|_{L\mathfrak{M}} \preceq \Lambda_{L\mathfrak{M}}$.

Let now $\|\cdot\|_{\mathfrak{M}}$ be an admissible norm on $\mathfrak{M} \in \mathcal{F}^R$. Then $L_a : \mathfrak{M}[\|\cdot\|_{\mathfrak{M}}] \to \mathfrak{A}[\|\cdot\|]$ is continuous, that is,

$$\|L_a x\| = \|ax\| \leqslant \gamma \|x\|_{\mathfrak{M}}, \quad x \in \mathfrak{M}$$

Then we can define a norm $\|\cdot\|^{\diamond}_{L\mathfrak{M}}$ on $L\mathfrak{M}$ by

$$\|a\|^{\diamond}_{L\mathfrak{M}} = \sup_{\|x\|_{\mathfrak{M}} \leqslant 1} \|ax\|. \tag{3.1}$$

Since the unit ball of $\mathfrak{M}[\|\cdot\|_{\mathfrak{M}}]$ is bounded in $\mathfrak{M}[\rho_{\mathfrak{M}}]$, it follows that $\|\cdot\|^{\diamond}_{L\mathfrak{M}}$ is admissible too. In the same way, we can define a new norm $\|\cdot\|^{\diamond\diamond}_{\mathfrak{M}}$ on $\mathfrak{M}$ by

$$\|x\|^{\diamond\diamond}_{\mathfrak{M}} = \sup_{\|a\|^{\diamond}_{L\mathfrak{M}} \leqslant 1} \|ax\|. \tag{3.2}$$

It is easily seen that $\|x\|^{\diamond\diamond}_{\mathfrak{M}} \leqslant \|x\|_{\mathfrak{M}}$, for every $x \in \mathfrak{M}$ and that $\|x\|^{\diamond\diamond}_{\mathfrak{M}}$ is admissible.

Proposition 3.4 *The following relations hold:*

- $\|x\|^{\diamond\diamond}_{\mathfrak{M}} \leqslant \|x\|_{\mathfrak{M}}, \ \forall\, x \in \mathfrak{M}$;
- $\|x\|^{\diamond\diamond\diamond}_{L\mathfrak{M}} = \|x\|^{\diamond}_{L\mathfrak{M}}, \ \forall\, x \in L\mathfrak{M}$.

This leads to a natural definition : the norm $\|\cdot\|_{\mathfrak{M}}$ on $\mathfrak{M}$ is said to be *reproducing* if $\|\cdot\|^{\diamond\diamond}_{\mathfrak{M}} \sim \|\cdot\|_{\mathfrak{M}}$

Proposition 3.5 *Let $\mathfrak{M} \in \mathcal{F}^R$ be endowed with an admissible norm $\|\cdot\|_{\mathfrak{M}}$. Assume that for every $a \in L\mathfrak{M}$, the map $L_a : x \in \mathfrak{M} \mapsto ax \in \mathfrak{A}$ is closed in $\mathfrak{A}$. Then $\mathfrak{M}[\|\cdot\|_{\mathfrak{M}}]$ is a Banach space if, and only if, the norm $\|\cdot\|_{\mathfrak{M}}$ is reproducing.*

Using these notions, we may formulate a new definition.

Definition 3.6 Let $\mathfrak{A}[\tau]$ be a locally convex partial *-algebra, and $\mathcal{I}^R$ a generating family. $\mathfrak{A}[\tau]$ is said to be a *normed partial *-algebra* if

(i) τ is a norm topology, with norm $\|\cdot\|$;

(ii) the involution $x \mapsto x^*$ is isometric on $\mathfrak{A}[\|\cdot\|]$;

(iii) each space $\mathfrak{M} \in \mathcal{I}^R$ is a Banach space for a reproducing norm $\|\cdot\|_{\mathfrak{M}}$.

$\mathfrak{A}[\tau]$ is called a *Banach partial *-algebra* if, in addition,

(iv) $\mathfrak{A}[\tau]$ is a Banach space.

We emphasize that completeness of $\mathfrak{A}[\|\cdot\|]$ may be relaxed (which leads to the definition of a normed partial *-algebra), but *not* that of the multiplier spaces $\mathfrak{M}[\|\cdot\|_{\mathfrak{M}}] \in \mathcal{I}^R$, where $\mathcal{I}^R$ is a generating family. Indeed, these spaces are completely determined by the partial multiplication (i.e., the set Γ). If one of them, say $\mathfrak{M}$, would be noncomplete, it could be embedded into its completion $\widetilde{\mathfrak{M}}$ w.r. to $\|\cdot\|_{\mathfrak{M}}$, but nothing guarantees that the latter is still contained in $\mathfrak{A}$, and thus there is *a priori* no way of extending the partial multiplication to $\widetilde{\mathfrak{M}}$! We refer to [12] or [13] for further details.

3.3 Examples of Banach or locally convex partial *-algebras

(i) L^p spaces on a finite interval

A standard example of an abelian partial *-algebra [7] is the space $L^1([0,1],dx)$, equipped with the partial multiplication:

$$f \in M(g) \ \Leftrightarrow \ \exists\, q \in [1,\infty] \ \text{ such that } f \in L^q,\ g \in L^{\bar{q}},\ 1/q + 1/\bar{q} = 1. \tag{3.3}$$

A similar structure may be given for every L^p.

Now we consider all spaces $L^p([0,1],dx)$ at once, that is, the chain $\mathcal{I} = \{L^p([0,1],dx), 1 \leqslant p \leqslant \infty\}$, with $L^p \subset L^q$, $p > q$. For $1 < p < \infty$, every space L^p is a reflexive Banach space with dual $L^{\bar{p}}$ $(1/p + 1/\bar{p} = 1)$. Notice that duality in the sense of Banach spaces coincides here with duality for the inner product of L^2, thanks to Hölder's inequality.

Now, being a chain, $\mathcal{I}$ is of course a lattice, albeit not a complete one. The lattice completion of $\mathcal{I}$, denoted $\mathcal{F}$, may be characterized explicity. Define the two spaces :

$$L^{p-} = \bigcap_{1 \leqslant q < p} L^q, \quad L^{p+} = \bigcup_{p < q \leqslant \infty} L^q.$$

Then, for $1 < p \leqslant \infty$, L^{p-}, with the projective topology, is a non-normable reflexive Fréchet space, with dual $L^{\bar{p}+}$. And for $1 \leqslant p < \infty$, L^{p+}, with the inductive topology, is a nonmetrizable complete DF-space, with dual $L^{\bar{p}-}$. Finally the following inclusions are strict:

$$L^{p+} \subset L^p \subset L^{p-} \subset L^{q+} \quad (1 < q < p < \infty), \tag{3.4}$$

all embeddings in (3.4) are continuous and have dense range. Then the complete lattice $\mathcal{F}$ generated by $\mathcal{I}$ is also a chain, obtained by replacing each L^p $(1 < p < \infty)$ by the corresponding triplet as in (3.4) and adding the two spaces $L^{\infty-}$ and L^{1+}:

$$L^\infty \subset L^{\infty-} \subset \ldots \subset L^{p+} \subset L^p \subset L^{p-} \subset \ldots \subset L^{1+} \subset L^1.$$

Now we turn to the partial *-algebra structure. The commutative partial multiplication on the space $L^1([0,1],dx)$ is defined as in (3.3), i.e., $\mathcal{I}$ is a generating family. Then it is easy to see that

$$ML^p = L^{\bar{p}}, \quad ML^{p-} = L^{\bar{p}+}, \quad ML^{p+} = L^{\bar{p}-}.$$

As for the multiplier topologies, we have that

- The topologies ρ_{L^p} and P_{L^p} are both equivalent to the L^p norm topology.
- The topologies $\rho_{L^{p-}}$ and $\mathrm{P}_{L^{p-}}$ are both equivalent to the Fréchet projective topology on L^{p-}.
- The topologies $\rho_{L^{p+}}$ and $\mathrm{P}_{L^{p+}}$ are both equivalent to the DF topology on L^{p+}.

For both $\mathcal{I}$ and $\mathcal{F}$, the smallest space is $L^\infty = ML^1$, and it is dense in all the other ones. The involution $f \mapsto \bar{f}$ is of course L^1-continuous. The multiplication is continuous from $L^\infty \times L^1$ into L^1. In fact it is not only separately, but even jointly continuous, and similarly from $L^p \times L^{\bar{p}}$ and from $L^{p-} \times L^{\bar{p}+}$ into L^1, thanks to Hölder's inequality and the fact that all topologies are either Fréchet or DF. In conclusion, the topological structure and the multiplier structure of $\mathcal{I}$ coincide, and we have a tight abelian Banach partial *-algebra [10].

Similarly, one may consider the scale : $\mathcal{I}_o = \{L^p([0,1],dx), 1 < p < \infty\}$ with largest space $L^{1+} = \bigcup_q L^q$. For the same reasons as before, $L^{1+}[0,1]$ is a tight abelian locally convex partial *-algebra. In addition, the chains $\mathcal{I}$ and $\mathcal{I}_o$ are partial inner product spaces [1, 2], and the latter structure coincides with the other two.

(ii) The spaces $L^p(\mathbb{R}, dx)$

We turn now to the spaces $L^p(\mathbb{R}, dx)$ on the whole line. The difference with the previous case is that these no longer form a chain, no two of them being comparable. We have only

$$L^p \cap L^q \subset L^s, \forall\, s \ \text{ such that } \ p < s < q.$$

Hence we have to take the lattice generated by $\mathcal{I} = \{L^p(\mathbb{R}, dx), 1 \leqslant p \leqslant \infty\}$, that we call $\mathcal{J}$. The extreme spaces of the lattice are, respectively:

$$V_J^{\#} = \bigcap_{1 \leqslant\, q \leqslant\, \infty} L^q, \quad \text{ and } \quad V_J = \bigcup_{1 \leqslant\, q \leqslant\, \infty} L^q = \sum_{1 \leqslant\, q \leqslant\, \infty} L^q.$$

Here too, the lattice structure allows to give to V_J the structure of a locally convex partial *-algebra. The lattice operations on $\mathcal{J}$ are those familiar in interpolation theory [14]:

- $L^p \wedge L^q = L^p \cap L^q$ is a Banach space, with the projective norm $\|f\|_{p \wedge q} = \|f\|_p + \|f\|_q$.
- $L^p \vee L^q = L^p + L^q$ is a Banach space, with the inductive norm $\|f\|_{p \vee q} = \inf\left(\|g\|_p + \|h\|_q\right)$, $f = g + h,\ g \in L^p,\ h \in L^q$.
- For $1 < p, q < \infty$, both spaces $L^p \wedge L^q$ and $L^p \vee L^q$ are reflexive and $(L^p \wedge L^q)' = L^{\bar{p}} \vee L^{\bar{q}}$.

Notice that the lattice $\mathcal{J}$ (depicted in Figure 1) is already obtained at the first generation: one has, for example, $L^{(r,s)} \wedge L^{(a,b)} = L^{(r \vee a, s \wedge b)}$. Furthermore, in the lattice $\mathcal{J}$, inclusion means continuous embedding with dense range. Thus, with the same partial multiplication (3.3), we obtain another tight locally convex partial *-algebra.

Two remarks are in order. First, here too, the lattice completion $\mathcal{F}$ of $\mathcal{J}$ and the multiplier spaces may be characterized explicitly [17]. Second, another structure of locally convex partial *-algebra may be given to the family $\mathcal{J}$ of spaces, simply replacing multiplication by convolution, with similar results [10, 12].

We note finally that the only difference between the two cases $\{L^p([0, 1])\}$ and $\{L^p(\mathbb{R})\}$ lies in the type of order obtained: a chain $\mathcal{I}$ (total order) or a partially ordered lattice $\mathcal{J}$.

(iii) Amalgam spaces

The lesson of the previous example is that an involutive lattice of (preferably reflexive) Banach spaces turns quite naturally into a (tight) locally convex partial *-algebra if it possesses a partial multiplication that satisfies a (generalized) Hölder inequality. A whole class of examples is given by the so-called *amalgam spaces*, first introduced by N. Wiener (see [19] for a review). The simplest ones are the spaces (L^p, ℓ^q), consisting of functions on $\mathbb{R}$ which are locally in L^p and have ℓ^q behavior at infinity, in the sense that the L^p norms over the intervals $(n, n + 1)$ form an ℓ^q sequence. For $1 < p, q < \infty$, the corresponding norm

$$\|f\|_{p,q} = \left\{ \sum_{n=-\infty}^{\infty} \left[\int_n^{n+1} |f(x)|^p dx \right]^{q/p} \right\}^{1/q}$$

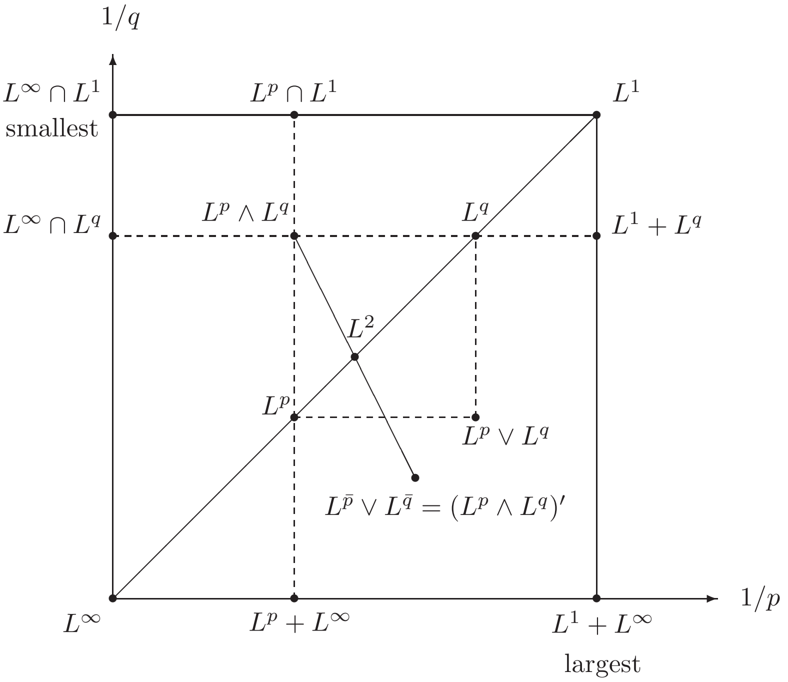

Figure 1: The lattice $\mathcal{J}$.

turns the space (L^p, ℓ^q) into a reflexive Banach space. The same is true for the obvious extensions to p and/or q equal to 1 or ∞. Notice that $(L^p, \ell^p) = L^p$. Once again, the set of all spaces (L^p, ℓ^q) may be partially ordered by inclusion, and it turns out to be a complete lattice. Thus one gets another locally convex partial *-algebra [10, 12].

*(iv) Locally convex partial *-algebras of operators*

A first example is the partial *-algebra of operators on a scale of Hilbert spaces, with the usual operator multiplication [2]. There are several cases to consider here, so we start with the simplest one, namely, a single triplet of Hilbert spaces, associated to a positive self-adjoint operator $S \geqslant 1$,

$$\mathcal{H}_1 \subset \mathcal{H} \subset \mathcal{H}_{-1}, \tag{3.5}$$

in the sense that $\mathcal{H}_1 = D(S)$, equipped with the graph norm $\|f\|_1 = \|Sf\|$ and the corresponding inner product $\langle f|g\rangle_1 = \langle Sf|Sg\rangle$, and $\mathcal{H}_{-1} = \mathcal{H}_1^\times$, the conjugate dual of $\mathcal{H}_1$, itself a Hilbert space. Now, for every $\alpha > 0$, S^α is also a positive self-adjoint operator and $S^\alpha \geqslant 1$. So we obtain a continuous family of Hilbert spaces $\{\mathcal{H}_\alpha, -1 \leqslant \alpha \leqslant 1\}$, where, for $\alpha \geqslant 0$, $\mathcal{H}_\alpha = D(S^\alpha)$, with the graph norm, and $\mathcal{H}_{-\alpha} = \mathcal{H}_\alpha^\times$. Actually, this family is a chain, since, for $0 < \alpha < \beta < 1$, one has

$$\mathcal{H}_1 \subset \mathcal{H}_\beta \subset \mathcal{H}_\alpha \subset \mathcal{H} \subset H_{-\alpha} \subset \mathcal{H}_{-\beta} \subset \mathcal{H}_{-1}, \tag{3.6}$$

and each embedding is continuous with dense range.

Now we consider the operators on the chain (3.6). Let $\mathcal{B}(\mathcal{H}_1, \mathcal{H}_{-1})$ be the Banach space of bounded operators from $\mathcal{H}_1$ into $\mathcal{H}_{-1}$ with its natural norm $\|\cdot\|_{1,-1}$. In $\mathcal{B}(\mathcal{H}_1, \mathcal{H}_{-1})$, we define an involution $A \mapsto A^*$ by the relation

$$< A^* f, g > = \overline{< Ag, f >}, \quad \forall\, f, g \in \mathcal{H}_{+1}\,,$$

where $< \cdot, \cdot >$ is the form that puts $\mathcal{H}_1$ and $\mathcal{H}_{-1}$ in conjugate duality.

If $\alpha, \beta \in (-1, 1)$, we can also consider the Banach space $\mathcal{B}(\mathcal{H}_\alpha, \mathcal{H}_\beta)$ of bounded operators from $\mathcal{H}_\alpha$ into $\mathcal{H}_\beta$ with its natural norm $\|\cdot\|_{\alpha,\beta}$. Because of (3.6), the restriction to $\mathcal{H}_1$ of an operator of $\mathcal{B}(\mathcal{H}_\alpha, \mathcal{H}_\beta)$ belongs to $\mathcal{B}(\mathcal{H}_1, \mathcal{H}_{-1})$. Therefore,

$$\mathcal{B}(\mathcal{H}_\alpha, \mathcal{H}_\beta) \ \subset \ \mathcal{B}(\mathcal{H}_1, \mathcal{H}_{-1}),\ \forall\, \alpha, \beta \in [-1, 1].$$

Moreover, $\mathcal{B}(\mathcal{H}_\alpha, \mathcal{H}_\beta)^* = \mathcal{B}(\mathcal{H}_{-\beta}, \mathcal{H}_{-\alpha})$ for every $\alpha, \beta \in [-1, 1]$.

Let us now define the partial multiplication in $\mathcal{B}(\mathcal{H}_1, \mathcal{H}_{-1})$. Let $X, Y \in \mathcal{B}(\mathcal{H}_1, \mathcal{H}_{-1})$. We say that $X \in L(Y)$ if there exist $\alpha, \beta, \gamma \in [-1, 1]$ such that $Y \in \mathcal{B}(\mathcal{H}_\alpha, \mathcal{H}_\beta)$ and $X \in \mathcal{B}(\mathcal{H}_\beta, \mathcal{H}_\gamma)$. In this case $X \cdot Y$, the usual composition of the maps X and Y, is well-defined and belongs to $\mathcal{B}(\mathcal{H}_\alpha, \mathcal{H}_\gamma) \subset \mathcal{B}(\mathcal{H}_1, \mathcal{H}_{-1})$. It easily seen that, if $X \cdot Y$ is well-defined, then $Y^* \cdot X^*$ is also well-defined and belongs to $\mathcal{B}(\mathcal{H}_{-\gamma}, \mathcal{H}_{-\alpha})$. Moreover $(X \cdot Y)^* = Y^* \cdot X^*$. As a result, $\mathcal{B}(\mathcal{H}_1, \mathcal{H}_{-1})$ is a partial *-algebra with respect to the $\cdot$ multiplication.

The next step is the identification of the spaces of multipliers. By the definition of multiplication just given, it follows that the family of spaces $\{\mathcal{B}(\mathcal{H}_\alpha, \mathcal{H}_\beta),\ -1 \leqslant \alpha, \beta \leqslant 1\}$ is a generating sublattice for the lattice of left (or right) multipliers. A small calculation then shows that, for every $\alpha, \beta \in [-1, 1]$, $L\mathcal{B}(\mathcal{H}_\alpha, \mathcal{H}_\beta) = \mathcal{B}(\mathcal{H}_\beta, \mathcal{H}_{-1})$ and $R\mathcal{B}(\mathcal{H}_\alpha, \mathcal{H}_\beta) = \mathcal{B}(\mathcal{H}_1, \mathcal{H}_\alpha)$. Furthermore, it is easy to see that the topology $\rho_{\mathcal{B}(\mathcal{H}_\alpha, \mathcal{H}_\beta)}$ is equivalent to the norm topology defined by $\| \cdot \|_{\alpha,\beta}$. It follows that the norm of each space $\mathcal{B}(\mathcal{H}_\alpha, \mathcal{H}_\beta)$ is reproducing. In conclusion, $\mathcal{B}(\mathcal{H}_1, \mathcal{H}_{-1})$ is a Banach partial *-algebra under the algebraic and topological structure defined above.

Moreover, we can here also ‘enrich’ the chain by introducing, for each $0 < \beta \leqslant 1$, the spaces

$$\mathcal{H}_{\beta^-} = \bigcap_{0<\alpha<\beta} \mathcal{H}_\alpha, \quad \mathcal{H}_{-\beta^+} = \bigcup_{0<\alpha<\beta} \mathcal{H}_{-\alpha}.$$

In general, $\mathcal{H}_{\beta^-}$ is a reflexive Fréchet space, whose dual is $\mathcal{H}_{-\beta^+}$, a reflexive DF-space. As in the example (i), the enriched chain is then the complete lattice generated by the chain $\{\mathcal{H}_\alpha\}$.

Starting from the construction just described, one can build several other examples. The first one consist in the discrete Hilbert scale built on the successive powers of the operator S,

$$\ldots \subset \mathcal{H}_2 \subset \mathcal{H}_1 \subset \mathcal{H} \subset \mathcal{H}_{-1} \subset \mathcal{H}_{-2} \subset \ldots, \tag{3.7}$$

where, for $n \in \mathbb{N}$, $\mathcal{H}_n = D(S^n)$, with the graph norm $\|f\|_n = \|S^n f\|$, and $\mathcal{H}_{-n} = \mathcal{H}_n^\times$,the conjugate dual of $\mathcal{H}_n$. From the discrete scale $\{\mathcal{H}_n,\, n \in \mathbb{Z}\}$, one can then build a continuous chain $\{\mathcal{H}_\alpha,\, \alpha \in \mathbb{R}\}$, by interpolation methods [14]. An example is the familiar chain of Sobolev spaces

$W^2_s(\mathbb{R})$, $s \in \mathbb{R}$, where $f \in W^2_s(\mathbb{R})$ if its Fourier transform $\widehat{f}$ satisfies the condition $(1+|.|^2)^{s/2}\,\widehat{f} \in L^2(\mathbb{R})$. Although the construction is similar, there is an essential difference, however. Whereas the operators on the bounded chain $\{\mathcal{H}_\alpha, -1 \leqslant \alpha \leqslant 1\}$ form a *Banach* partial *-algebra, in the case on an infinite scale one gets only a *locally convex* partial *-algebra. Actually the same statement is true for the 'open' chain $\{\mathcal{H}_\alpha, -1 < \alpha < 1\}$, exactly as with the L^p spaces discussed above: one gets a Banach partial *-algebra only if the extreme elements of the family are themselves Hilbert spaces.

Interestingly enough, the same structure is obtained if one considers operators on a *lattice* of Hilbert spaces, instead of a scale, or for that matter, operators on a partial inner product space [1, 2].

A completely different type of example is given by partial *-algebras of closable operators in a Hilbert space. From now on, we will mostly concentrate on this class. We refer to [9] for further details and the original references.

4 Partial *-algebras of closable operators

4.1 Basic definitions and properties

Let $\mathcal{H}$ be a complex Hilbert space and $\mathcal{D}$ a dense subspace of $\mathcal{H}$. We denote by $\mathcal{L}^\dagger(\mathcal{D},\mathcal{H})$ the set of all (closable) linear operators X such that $\mathcal{D}(X) = \mathcal{D}$, $\mathcal{D}(X^*) \supseteq \mathcal{D}$. The set $\mathcal{L}^\dagger(\mathcal{D},\mathcal{H})$ is a partial *-algebra with respect to the following operations: the usual sum $X_1 + X_2$, the scalar multiplication λX, the involution $X \mapsto X^\dagger = X^* \lceil \mathcal{D}$ and the *(weak)* partial multiplication $X_1 \,\square\, X_2 = {X_1}^{\dagger *} X_2$, defined whenever X_2 is a weak right multiplier of X_1 (equivalently, X_1 is a weak left multiplier of X_2), that is, iff $X_2\mathcal{D} \subset \mathcal{D}({X_1}^{\dagger *})$ *and* ${X_1}^*\mathcal{D} \subset \mathcal{D}({X_2}^*)$ (we write $X_2 \in R^{\rm w}(X_1)$ or $X_1 \in L^{\rm w}(X_2)$). When we regard $\mathcal{L}^\dagger(\mathcal{D},\mathcal{H})$ as a partial *-algebra with these operations, we denote it by $\mathcal{L}^\dagger_{\rm w}(\mathcal{D},\mathcal{H})$.

A *partial O*-algebra* on $\mathcal{D}$ is a *-subalgebra $\mathfrak{M}$ of $\mathcal{L}^\dagger_{\rm w}(\mathcal{D},\mathcal{H})$, that is, $\mathfrak{M}$ is a subspace of $\mathcal{L}^\dagger_{\rm w}(\mathcal{D},\mathcal{H})$, containing the identity and such that $X^\dagger \in \mathfrak{M}$ whenever $X \in \mathfrak{M}$ and $X_1 \,\square\, X_2 \in \mathfrak{M}$ for any $X_1, X_2 \in \mathfrak{M}$ such that $X_2 \in R^{\rm w}(X_1)$. Thus $\mathcal{L}^\dagger_{\rm w}(\mathcal{D},\mathcal{H})$ itself is the largest partial O*-algebra on the domain $\mathcal{D}$.

On the space $\mathcal{L}^\dagger(\mathcal{D},\mathcal{H})$ we will consider the *strong** topology τ_{s^*}, which is generated by the family of seminorms : ${p_\xi}^*(X) = \|X\xi\| + \|X^\dagger\xi\|, \xi \in \mathcal{D}$. The space $\mathcal{L}^\dagger(\mathcal{D},\mathcal{H})$ is complete for τ_{s^*}. For $\mathfrak{N} \subset \mathcal{L}^\dagger(\mathcal{D},\mathcal{H})$, we denote by $[\mathfrak{N}]^{s^*}$ the τ_{s^*}-closure of $\mathfrak{N}$.

We also need the *weak* topology $\tau_{\rm w}$ on $\mathcal{L}^\dagger(\mathcal{D},\mathcal{H})$, which is generated by the family of seminorms $p_{f,g}(X) = |\langle f|Xg\rangle|$, $f, g \in \mathcal{D}$, and the *quasi-uniform* topology, τ_*, defined by the set of seminorms $p_{\mathcal{N}}(X) = \sup_{f\in\mathcal{N}}(\|Xf\| + \|X^\dagger f\|)$, where $\mathcal{N}$ is a bounded subset of $\mathcal{D}$, equipped with the projective topology determined by $\mathcal{L}^\dagger(\mathcal{D},\mathcal{H})$.

If we restrict ourselves to those operators in $\mathcal{L}^\dagger(\mathcal{D},\mathcal{H})$ that, together with their adjoint, leave the domain $\mathcal{D}$ invariant, we obtain a *-algebra, namely $\mathcal{L}^\dagger(\mathcal{D}) = \{A \in \mathcal{L}^\dagger(\mathcal{D},\mathcal{H}); A\mathcal{D} \subset$

$\mathcal{D}$ and $A^*\mathcal{D} \subset \mathcal{D}\}$. Then an *O*-algebra* is defined as a *-subalgebra of $\mathcal{L}^\dagger(\mathcal{D})$; thus $\mathcal{L}^\dagger(\mathcal{D})$ is the maximal O*-algebra contained in $\mathcal{L}^\dagger(\mathcal{D},\mathcal{H})$ and it is τ_{s^*}-dense in $\mathcal{L}^\dagger(\mathcal{D},\mathcal{H})$, i.e., $\mathcal{L}^\dagger(\mathcal{D},\mathcal{H}) = [\mathcal{L}^\dagger(\mathcal{D})]^{s^*}$. Clearly an O*-algebra is a particular case of a partial O*-algebra (see [21] for a comprehensive study of partial O*-algebras).

Given a partial O*-algebra $\mathfrak{M}$, we define *internal multipliers* as $R(X) = R^{\rm w}(X) \cap \mathfrak{M}$ and $L(X) = L^{\rm w}(X) \cap \mathfrak{M}$. Then the universal right multipliers of $\mathfrak{M}$ are the elements of the set:

$$R\mathfrak{M} = R^{\rm w}(\mathfrak{M}) \cap \mathfrak{M} = \{Y \in \mathfrak{M};\ X \,\square\, Y \ \text{ is well-defined, } \forall X \in \mathfrak{M}\}.$$

A †-invariant subset $\mathfrak{N}$ of $\mathcal{L}^\dagger(\mathcal{D},\mathcal{H})$ is called *fully closed* if $\mathcal{D} = \widehat{\mathcal{D}}(\mathfrak{N}) \equiv \bigcap_{X\in\mathfrak{N}} \mathcal{D}(\overline{X})$. If $\mathfrak{N}$ is not fully closed, its full closure is the smallest fully closed set that contains it, that is, $\widehat{\mathfrak{N}} = \{\hat{\iota}(X) \equiv \overline{X}\lceil\widehat{\mathcal{D}}(\mathfrak{N});\ X \in \mathfrak{N}\}$. Let $\mathfrak{M}$ be a partial O*-algebra. If it is not fully closed, it may be embeddded into its full closure $\widehat{\mathfrak{M}} = \hat{\iota}(\mathfrak{M})$, which is a fully closed partial O*-algebra on the domain $\widehat{\mathcal{D}}(\mathfrak{M})$, isomorphic to $\mathfrak{M}$. Thus one may always restrict the analysis to fully closed partial O*-algebras without loss of generality. On the other hand, a partial O*-algebra $\mathfrak{M}$ is called *self-adjoint* if $\mathcal{D} = \mathcal{D}^*(\mathfrak{M}) \equiv \bigcap_{X\in\mathfrak{M}} \mathcal{D}(X^*)$, and this *is* a strong restriction.

Given a †-invariant subset $\mathfrak{N}$ of $\mathcal{L}^\dagger(\mathcal{D},\mathcal{H})$, we define, as usual, its *weak unbounded commutant*:

$$\mathfrak{N}'_\sigma = \{Y \in \mathcal{L}^\dagger(\mathcal{D},\mathcal{H}); (X\xi|Y\eta) = (Y^\dagger\xi|X^\dagger\eta) \text{ for each } \xi,\eta \in \mathcal{D} \text{ and } X \in \mathfrak{N}\} \qquad (4.1)$$

and its *weak bounded commutant*:

$$\mathfrak{N}'_{\rm w} = \{C \in \mathcal{B}(\mathcal{H}); (CX\xi|\eta) = (C\xi|X^\dagger\eta) \text{ for each } \xi,\eta \in \mathcal{D} \text{ and } X \in \mathfrak{N}\}. \qquad (4.2)$$

The restriction to $\mathcal{D}$ of $\mathfrak{N}'_{\rm w}$ is the bounded part of $\mathfrak{N}'_\sigma$. Both $\mathfrak{N}'_\sigma$ and $\mathfrak{N}'_{\rm w}$ are weakly closed, †-invariant subspaces, but not necessarily algebras.

As for *bicommutant*, we consider the weak unbounded one, namely $\mathfrak{N}''_{{\rm w}\sigma} = (\mathfrak{N}'_{\rm w})'_\sigma$. Its bounded part is the (restriction to $\mathcal{D}$ of) $(\mathfrak{N}'_{\rm w})'$, where $\mathcal{B}'$ denotes the usual bounded commutant of a subset $\mathcal{B} \subset \mathcal{B}(\mathcal{H})$. We note the relation $(\mathfrak{N}''_{{\rm w}\sigma})''_{{\rm w}\sigma} = \mathfrak{N}''_{{\rm w}\sigma}$ and remark that $\mathfrak{N}''_{{\rm w}\sigma}$ is fully closed whenever $\mathfrak{N}$ is, because of the obvious inclusions $\mathcal{D} \subset \widehat{\mathcal{D}}(\mathfrak{N}''_{{\rm w}\sigma}) \subset \widehat{\mathcal{D}}(\mathfrak{N})$. The crucial fact is that, for any †-invariant subset $\mathfrak{N}$ of $\mathcal{L}^\dagger(\mathcal{D},\mathcal{H})$, $\mathfrak{N}'_{\rm w}$ is a von Neumann algebra if, and only if, $\mathfrak{N}''_{{\rm w}\sigma} = \left[(\mathfrak{N}'_{\rm w})'\lceil\mathcal{D}\right]^{s^*}$.

A partial O*-algebra $\mathfrak{M}$ on $\mathcal{D}$ is said to be a *partial GW*-algebra* if it is fully closed and satisfies the two conditions $\mathfrak{M}'_{\rm w}\mathcal{D} = \mathcal{D}$ and $\mathfrak{M}''_{{\rm w}\sigma} = \mathfrak{M}$ (notice the analogy with the usual condition $\mathfrak{M}'' = \mathfrak{M}$ defining a von Neumann algebra). In that case, $\mathfrak{M}'_{\rm w}$ is a von Neumann algebra, the (closure of the) bounded part of $\mathfrak{M}$ is also a von Neumann algebra, namely $\mathfrak{M}_o \equiv (\mathfrak{M}'_{\rm w})'$, and $\mathfrak{M} = \left[(\mathfrak{M}'_{\rm w})'\lceil\mathcal{D}\right]^{s^*}$. The good properties of partial GW*-algebras stem precisely from the fact that they contain a τ_{s^*}-dense subset of bounded operators.

The easiest way of constructing a partial GW*-algebra is to take a bicommutant. Indeed, if $\mathfrak{N}$ is a fully closed †-invariant subset of $\mathcal{L}^\dagger(\mathcal{D},\mathcal{H})$, then $\mathfrak{N}''_{{\rm w}\sigma}$ is a partial GW*-algebra on $\mathcal{D}$ iff

$\mathfrak{N}'_{\rm w}\mathcal{D} = \mathcal{D}$. On the other hand, if $\mathfrak{M}$ is a partial O*-algebra on $\mathcal{D}$ (not necessarily fully closed), such that $\mathfrak{M}'_{\rm w}\mathcal{D} = \mathcal{D}$ and $\mathfrak{M}''_{{\rm w}\sigma} = \mathfrak{M}$, then $\widehat{\mathfrak{M}}$ is a partial GW*-algebra on $\widehat{\mathcal{D}}(\mathfrak{M})$.

As a last point, we may ask the question whether a partial O*-algebra is a (tight) locally convex partial *-algebra. The answer, of course, depends on which topology τ one chooses, and many different ones are available, the strong* τ_{s^*}, the quasi-uniform τ_*, the weak $\tau_{\rm w}$, etc. We will not enter into the technical details, for lack of space, but only indicate a few general results. First, if $\mathcal{L}^\dagger_{\rm w}(\mathcal{D}, \mathcal{H})$ is self-adjoint, then it is a locally convex partial *-algebra for these three topologies, and it is complete for τ_{s^*} and τ_*. More generally, any self-adjoint partial O*-algebra $\mathfrak{M}$ is a locally convex partial *-algebra for the weak topology $\tau_{\rm w}$, and the same is true for τ_* if $R\mathfrak{M}$ contains only bounded operators. In all cases, tightness is open.

4.2 *-Automorphisms of partial O*-algebras

In the algebraic formulation of quantum theories, the observables of a physical system are represented by hermitian elements of a certain *-algebra $\mathfrak{A}$ and states by positive linear functionals on $\mathfrak{A}$. Then a symmetry of the system is realized by a *-automorphism σ of $\mathfrak{A}$, and a one parameter symmetry group by a *-automorphism group σ_t ($t \in \mathbb{R}$) of $\mathfrak{A}$. Given a state, the Gel'fand–Naĭmark–Segal (GNS) construction yields a representation π of $\mathfrak{A}$ by bounded operators in a Hilbert space $\mathcal{H}_\pi$ and a *-automorphism σ^π (resp.*-automorphism group σ^π_t) of $\pi(\mathfrak{A})$. Then the question is whether σ^π is *spatial*, that is, whether there exists a unitary operator U in $\mathcal{H}_\pi$ such that $\sigma^\pi(A) = U^* A U$, for every $A \in \pi(\mathfrak{A})$. Even more interesting is the case where U itself can be taken in $\pi(\mathfrak{A})$, i.e., the automorphism is *inner*. For a one parameter group σ_t, spatiality means that the automorphism group σ^π_t is unitarily implemented, i.e., $\sigma^\pi_t(A) = e^{-iHt} A e^{iHt}$, where H is a self-adjoint operator. In particular, if the automorphism is inner, this means that $H \in \pi(\mathfrak{A})''$ (or H is affiliated to $\pi(\mathfrak{A})''$), in other words that the operator H is an observable. For instance, if σ_t represents the time evolution of the system, then 'σ^π_t is inner' means that the Hamiltonian exists as an observable in the (GNS) representation at hand [15, 18, 20].

Now, if one decides to describe the set of observables of a given physical system by some partial *-algebra, in particular, a partial O*-algebra, one must generalize to that context the notion of *-automorphism, and of spatiality as well.

Let $\mathfrak{M}$ be a partial O*-algebra on $\mathcal{D}$, obtained, for instance, from the partial *-algebra of observables by a GNS construction (the latter indeed extends to partial *-algebras, as we shall see in Section 5.2). According to the general definition, a **-automorphism* of $\mathfrak{M}$ is a linear bijection $\sigma : \mathfrak{M} \to \mathfrak{M}$ such that (i) $\sigma(X^\dagger) = \sigma(X)^\dagger, \forall X \in \mathfrak{M}$; (ii) $\sigma(Y) \in R^{\rm w}(\sigma(X))$ iff $Y \in R^{\rm w}(X)$ and then $\sigma(X \,\Box\, Y) = \sigma(X) \,\Box\, \sigma(Y)$ and (iii) the same relations hold for σ^{-1}. It follows that $\sigma(\mathfrak{M}) = \mathfrak{M}$ and $\sigma(R\mathfrak{M}) = R\mathfrak{M}$. The *-automorphism σ is *spatial* if there exists a unitary operator $U \in \mathcal{H}$ such that $U_o \equiv U{\upharpoonright}\mathcal{D} \in R^{\rm w}(\mathfrak{M})$ and $\sigma(X) = U_o{}^*(X \,\Box\, U_o), \forall\, X \in \mathfrak{M}$. It is *inner* if in addition $U_o \in \mathfrak{M}$, i.e., $U_o \in R\mathfrak{M}$. Notice that, if $\mathfrak{M}$ is self-adjoint and σ is spatial, then

$U_o \in \mathcal{L}^\dagger(\mathcal{D})$ and $\sigma(X) = {U_o}^* X U_o, \forall\, X \in \mathfrak{M}$.

Armed with these seemingly natural definitions, one may study under what conditions a *-automorphism is spatial or inner, and also the structure of the corresponding one-parameter groups of *-automorphisms, both globally and at the infinitesimal level (using the notion of derivation). As expected, one gets significant results only if one assumes that $\mathfrak{M}$ is a partial GW*-algebra, because then many known results on *-automorphisms of von Neumann algebras will be lifted from the bounded part of $\mathfrak{M}$ to $\mathfrak{M}$ itself [8]. Indeed, if $\mathfrak{M}$ is a partial GW*-algebra on the domain $\mathcal{D}$, its bounded part $\mathfrak{M}_b = \{X \in \mathfrak{M}; \overline{X} \in \mathcal{B}(\mathcal{H})\}$ is the restriction to $\mathcal{D}$ of the von Neumann algebra $\overline{\mathfrak{M}_b} = \{\overline{X}; X \in \mathfrak{M}_b\}$, with commutant $\overline{\mathfrak{M}_b}' = \mathfrak{M}'_{\rm w}$. The key observation is that every *-automorphism σ of $\mathfrak{M}$ induces a *-automorphism σ_b of the von Neumann algebra $\overline{\mathfrak{M}_b}$, by the simple relation $\sigma_b(\overline{A}) = \overline{\sigma(A)},\ A \in \mathfrak{M}_b$. This is the crucial link, which allows one to control in a rather complete way the properties of *-automorphisms and *-automorphism groups on partial O*-algebras. For lack of space, we will stop here and refer the interested reader to [8] or [12].

4.3 Tomita–Takesaki theory of modular automorphisms

In the standard approach to quantum statistical mechanics, the equilibrium states of a physical system (Gibbs states) are described by the so-called KMS states on the observable algebra $\mathfrak{A}$, and these states are derived with the celebrated Tomita–Takesaki theory of modular automorphisms. Once again, the obvious question is, what happens if one starts from a partial O*-algebra $\mathfrak{M}$ as observable "algebra"? It turns out that the Tomita–Takesaki theory can be extended to that case, but the construction is rather involved. Hence we will give here only a brief, nontechnical summary. The full story may be found in [12].

The key notion (due to Tomita) for answering the question is that of a *generalized vector* on $\mathfrak{M}$. By this, one means a map $\lambda : \mathfrak{M} \to \mathcal{H}$, defined on a domain $\mathcal{D}(\lambda)$, such that there exists a subspace $B(\lambda)$ of $\mathfrak{M}$ with the following properties:

(i) $\mathcal{D}(\lambda) =$ linear span $\{Y \,\square\, X; X \in B(\lambda), Y \in L(X)\}$;

(ii) λ is linear on $\mathcal{D}(\lambda)$;

(iii) $\lambda(B(\lambda)) \subset \mathcal{D}$;

(iv) $\lambda(Y \,\square\, X) = Y\lambda(X)$ for every $X \in B(\lambda)$ and $Y \in L(X)$.

Such a subspace $B(\lambda)$ is called a *core* for λ, and every core $B(\lambda)$ can be embedded in a maximal core $B_M(\lambda)$.

Then, under suitable conditions (coded into the expression'$(\mathfrak{M}, \lambda)$ is a cyclic system'), one defines the *commutant* $\lambda^c : K\lambda(X) = X\lambda^c(K)$, $K \in \mathfrak{M}'_{\rm w}$, and it turns out that λ^c is a generalized vector for the von Neumann algebra $\mathfrak{M}'_{\rm w}$. Under similar conditions (one says, '$(\mathfrak{M}, \lambda, \lambda^c)$ is a cyclic and separating system'), one defines the *bicommutant* $\lambda^{cc} : A\lambda^c(K) = K\lambda^{cc}(A)$, $A \in (\mathfrak{M}'_{\rm w})'$, and again λ^{cc} is a generalized vector for the von Neumann algebra $(\mathfrak{M}'_{\rm w})'$.

At this stage, one defines the modular involutions (on suitable domains) :

$$\begin{aligned} S_\lambda &: \lambda(X) \mapsto \lambda(X^\dagger) \\ S_\lambda^{cc} &: \lambda^{cc}(A) \mapsto \lambda^{cc}(A^*). \end{aligned}$$

As in the standard von Neumann case, one shows that S_λ and $S_{\lambda^{cc}}$ are closable conjugate linear operators in $\mathcal{H}$. Consider then the polar decomposition of the respective closures (denoted by the same symbol, for simplicity), $S_\lambda = J_\lambda \Delta_\lambda^{1/2}$ and $S_{\lambda^{cc}} = J_{\lambda^{cc}} \Delta_{\lambda^{cc}}^{1/2}$. Using these and a number of technical conditions, one introduces the key notion of *standard* generalized vector, in terms of which the fundamental theorem may be stated as follows.

Theorem 4.1 *Let λ be a standard generalized vector for a partial O*-algebra $\mathfrak{M}$. Then :*

(1) $S_\lambda = S_{\lambda^{cc}}$, and thus $J_\lambda = J_{\lambda^{cc}}$ and $\Delta_\lambda = \Delta_{\lambda^{cc}}$.

*(2) Define $\sigma_t^\lambda(X) = \Delta_\lambda^{it} X \Delta_\lambda^{-it}$, $X \in \mathfrak{M}$, $t \in \mathbb{R}$. Then $\{\sigma_t^\lambda\}_{t\in\mathbb{R}}$ is a one-parameter group of *-automorphisms of $\mathfrak{M}$.*

(3) λ satisfies the KMS condition with respect to $\{\sigma_t^\lambda\}_{t\in\mathbb{R}}$, i.e., $\forall\, X, Y \in B_M(\lambda)^\dagger \cap B_M(\lambda)$, there exists a function $f_{X,Y}$, bounded and continuous in the strip $-1 \leqslant \mathrm{Im} z \leqslant 0$ and analytic in the interior, such that, for all $t \in \mathbb{R}$,

$$\begin{aligned} f_{X,Y}(t) &= (\lambda(\sigma_t^\lambda(X)) | \lambda(Y)), \\ f_{X,Y}(t-i) &= (\lambda(Y^\dagger) | \lambda(\sigma_t^\lambda(X^\dagger))). \end{aligned}$$

The next step in the theory is to relax somewhat the assumptions, with help of the weaker notion of *modular* generalized vector, then to consider the particular case where the partial *-algebra is a partial GW*-algebra. Finally one can prove a generalized Connes cocycle theorem, that allows to compare two generalized vectors on the same partial GW*-algebra.

In summary, essentially the complete Tomita–Takesaki theory extends to partial *-algebras, provided the appropriate notions are identified.

5 Representation theory

5.1 Generalities

A *-representation of a partial *-algebra $\mathfrak{A}$ is a *-homomorphism of $\mathfrak{A}$ into $\mathcal{L}^\dagger_{\mathrm{w}}(\mathcal{D}, \mathcal{H})$, for some pair $\mathcal{D} \subset \mathcal{H}$, that is, a linear map $\pi : \mathfrak{A} \to \mathcal{L}^\dagger_{\mathrm{w}}(\mathcal{D}, \mathcal{H})$ such that : (i) $\pi(x^*) = \pi(x)^\dagger$ for every $x \in \mathfrak{A}$; (ii) $x \in L(y)$ in $\mathfrak{A}$ implies $\pi(x) \in L^{\mathrm{w}}(\pi(y))$ and $\pi(x) \,\Box\, \pi(y) = \pi(xy)$.

Let π be a *-representation of a partial *-algebra $\mathfrak{A}$. It is called *fully closed* if $\pi(\mathfrak{A})$ is fully closed. In any case, a *-representation π can always be extended to a fully closed *-representation $\widehat{\pi}(\mathfrak{A})$, namely, $\widehat{\pi}(x) = \widehat{\pi(x)}, x \in \mathfrak{A}$, on the domain $\mathcal{D}(\widehat{\pi}) = \widehat{\mathcal{D}}(\pi(\mathfrak{A}))$.

Next we define the weak commutants of a *-representation of a partial *-algebra. Besides the usual weak bounded commutant $\pi(\mathfrak{A})'_{\rm w}$ of $\pi(\mathfrak{A})$, as defined in (4.2), we introduce a new one, called *quasi-weak*, which takes explicitly into account the possible lack of associativity :

$$\begin{aligned}\mathcal{C}_{\rm qw}(\pi) \;\;=\;\; \{C \in \pi(\mathfrak{A})'_{\rm w}; (C\pi(x_1{}^*)\xi|\pi(x_2)\eta) = (C\xi|\pi(x_1x_2)\eta),\ \text{for all } x_1, x_2 \in \mathfrak{A} \\ \text{such that } x_1 \in L(x_2) \text{ and all } \xi, \eta \in \mathcal{D}(\pi)\}. \qquad (5.1)\end{aligned}$$

$\mathcal{C}_{\rm qw}(\pi)$ is a weakly closed *-invariant subspace of $\mathcal{B}(\mathcal{H})$, contained in $\pi(\mathfrak{A})'_{\rm w}$, and, moreover, $\mathcal{C}_{\rm qw}(\widehat{\pi}) = \mathcal{C}_{\rm qw}(\pi)$. Thus we have now two possible definitions of irreducibility, namely:

- π is *irreducible* iff $C_{\rm qw}(\pi) = \pi(\mathfrak{A})'_{\rm w} = \mathbb{C}I$;
- π is *weakly irreducible* iff $C_{\rm qw}(\pi) = \mathbb{C}I$.

5.2 The GNS construction

As always, the crucial question is how to build concrete representations. For *-algebras, the Gel'fand–Naĭmark–Segal (GNS) construction is usually the answer [16]. In order to extend it to partial *-algebras, we must first have an appropriate notion of state. In the case of a *-algebra $\mathfrak{A}$, a state is a normalized positive linear form on $\mathfrak{A}$. But the positivity condition requires the existence of products a^*a, which need not exist in a partial *-algebra. The alternative, of course, is to use *sesquilinear* forms. However, for a *-algebra $\mathfrak{A}$, the GNS construction works only if the starting sesquilinear form ϕ on $\mathfrak{A} \times \mathfrak{A}$ is *invariant*, in the sense that $\phi(x^*y, z) = \phi(y, xz)$, for all $x, y, z \in \mathfrak{A}$. Clearly this definition is inapplicable for a partial *-algebra, since the products x^*y and xz need not exist. An obvious solution is to impose this relation for $y, z \in R\mathfrak{A}$ only, and this gives us a good hint.

Let $\mathfrak{A}$ be a partial *-algebra, and let φ be sesquilinear form on $D(\varphi) \times D(\varphi)$, where $D(\varphi)$ a subspace of $\mathfrak{A}$. The form φ is said to be *positive* if $\varphi(x, x) \geqslant 0, \forall\, x \in D(\varphi)$. Then we have:

$$\varphi(x, y) \;\;=\;\; \overline{\varphi(y, x))},\ \forall\, x, y \in D(\varphi) \qquad (5.2)$$

$$|\varphi(x, y)|^2 \;\;\leqslant\;\; \varphi(x, x)\,\varphi(y, y),\ \forall\, x, y \in D(\varphi), \qquad (5.3)$$

and hence

$$N_\varphi \equiv \{x \in D(\varphi); \varphi(x, x) = 0\} = \{x \in D(\varphi); \varphi(x, y) = 0 \text{ for all } y \in D(\varphi)\ \}, \qquad (5.4)$$

and so N_φ is a subspace of $\mathfrak{A}$. For each $x \in D(\varphi)$, we denote by $\lambda_\varphi(x)$ the element of $D(\varphi)/N_\varphi$ which contains x, and define an inner product $(.|.)$ on $\lambda_\varphi(D(\varphi)) = D(\varphi)/N_\varphi$ by

$$(\lambda_\varphi(x)|\lambda_\varphi(y)) = \varphi(x, y),\ x, y \in D(\varphi). \qquad (5.5)$$

We denote by $\mathcal{H}_\varphi$ be the Hilbert space obtained by the completion of the pre-Hilbert space $\lambda_\varphi(D(\varphi))$. We are now ready to introduce our notion of invariance.

Definition 5.1 – Let φ be a positive sesquilinear form on $D(\varphi) \times D(\varphi)$. A *core* for φ is a subspace $B(\varphi)$ of $D(\varphi)$ such that

- $B(\varphi) \subset R\mathfrak{A}$;
- $\{ax; a \in \mathfrak{A}, x \in B(\varphi)\} \subset \mathcal{D}(\varphi)$;
- $\lambda_\varphi(B(\varphi))$ is dense in $\mathcal{H}_\varphi$;
- $\varphi(ax, y) = \varphi(x, a^* y), \quad \forall\, a \in \mathfrak{A}, \forall\, x, y \in B(\varphi)$;
- $\varphi(a^* x, by) = \varphi(x, (ab)y), \quad \forall\, a \in L(b), \forall\, x, y \in B(\varphi)$.

Obviously, the last condition takes care of the possible non-associativity of $\mathfrak{A}$.

We denote by $\mathcal{B}_\varphi$ the set of all cores $B(\varphi)$ for φ and call *biweight* on $\mathfrak{A}$ a positive sesquilinear form φ on $D(\varphi) \times D(\varphi)$ such that $\mathcal{B}_\varphi$ is nonempty [11]. If $\mathfrak{A}$ has a unit e, we say that a biweight φ, with core $B(\varphi)$, is a *state* if $\varphi(e, e) = 1$.

This notion is biweight is precisely the one that allows the GNS construction on a partial *-algebra, as shown in the next proposition.

Proposition 5.2 *Let φ be a biweight on $\mathfrak{A}$ with a core $B(\varphi)$. Put*

$$\pi^\circ_\varphi(a)\lambda_\varphi(x) = \lambda_\varphi(ax), \quad a \in \mathfrak{A}, x \in B(\varphi).$$

*Then π°_φ is a *-representation of $\mathfrak{A}$ into $\mathcal{L}^\dagger_{\mathrm{w}}(\lambda_\varphi(B(\varphi)), \mathcal{H}_\varphi)$.*

Denote by π^B_φ the closure of π°_φ. We call the triple $(\pi^B_\varphi, \lambda_\varphi, \mathcal{H}_\varphi)$ the *GNS construction* for the biweight φ on $\mathfrak{A}$ with core $B(\varphi)$.

In view of this result, it is natural to ask whether one can characterize the GNS construction in terms of a core. Some care is needed, since it might happen that $\pi^{B_1}_\varphi = \pi^{B_2}_\varphi$ with $B_1(\varphi) \neq B_2(\varphi)$. The key observation is that the set of all cores giving the same GNS representation, $\{B_1(\varphi) \in \mathcal{B}_\varphi : \pi^{B_1}_\varphi = \pi^B_\varphi\}$, has a maximal element, namely

$$\begin{aligned} B_L(\varphi) = \{x \in D(\varphi) \cap R\mathfrak{A} \,;\, \lambda_\varphi(x) \in D(\pi^B_\varphi), \text{ and} \\ ax \in D(\varphi) \,\text{and}\, \lambda_\varphi(ax) = \pi^B_\varphi(a)\lambda_\varphi(x) \text{ for all } a \in \mathfrak{A}\}. \end{aligned}$$

Thus we say that $B(\varphi)$ is *GNS-maximal* whenever $B(\varphi) = B_L(\varphi)$. Then one gets a unique characterization:

Proposition 5.3 *Let φ be a biweight on $\mathfrak{A}$ and $B_1(\varphi), B_2(\varphi)$ two GNS-maximal cores for φ. Then :*

- $\pi^{B_1}_\varphi \subset \pi^{B_2}_\varphi \;\Leftrightarrow\; B_1(\varphi) \subset B_2(\varphi)$;
- $\pi^{B_1}_\varphi = \pi^{B_2}_\varphi \;\Leftrightarrow\; B_1(\varphi) = B_2(\varphi)$.

We may now define the appropriate notion of pure state. Let $\mathfrak{A}$ be a partial *-algebra with unit e and let φ be a state on $\mathfrak{A}$ with core $B(\varphi)$. We say that the state φ is *pure* if it cannot be decomposed into a convex combination of two states φ_1, φ_2 *with the same core* as φ :

$$\varphi \neq \lambda\varphi_1 + (1-\lambda)\varphi_2, \quad 0 < \lambda < 1, \;\; \varphi_1, \varphi_2 \text{ states with core } B(\varphi).$$

The interest of this concept is that the equivalence between the purity of a state φ and the irreducibility of its GNS representation $\pi_\varphi^{\mathfrak{B}}$ extends to partial *-algebras, essentially with the same proof:

Proposition 5.4 *Let $\mathfrak{A}$ be a partial *-algebra with unit, φ a state on $\mathfrak{A}$, with core $B(\varphi)$. Then the GNS representation π_φ^B is weakly irreducible, i.e., $C_{\rm qw}(\pi_\varphi^B) = \mathbb{C}I$, iff φ is a pure state.*

Actually, biweights seem a crucial concept, for they allow the extension to partial *-algebras of several classical results from von Neumann algebra theory [12]. For instance,

- The Radon–Nikodým theorem, concerning the relative domination of one biweight over another one;
- The Lebesgue decomposition theorem, into an absolutely continuous and a singular biweight;
- The definition of standard biweights on a partial GW*-algebra leads to a further generalization of the Tomita-Takesaki theory of modular automorphism groups.

6 Epilogue

The conclusion of this rapid survey is that the theory of partial *-algebras has reached after twenty years a reasonable stage of maturity. Many nontrivial examples have been studied, both abelian and nonabelian, although no classification has been made so far. The representation theory is well under control. In particular, many standard results extend to partial *-algebras, such as the GNS construction or various structure properties. Two offshoots, in particular, have undergone a rapid development, namely CQ*-algebras and partial O*-algebras. The latter, and among them partial GW*-algebras, are a far reaching generalization of *-algebras of operators, both bounded and unbounded. Their structure is quite complex, yet a substantial body of information is available. Besides the representation theory associated to various notions of generalized vectors and weights, progress has been achieved also in the study of *-automorphisms and *-derivations, in particular the spatial theory.

These last results point toward the most promising direction of research, namely the study of dynamical systems based on partial O*-algebras. In view of the results obtained so far, it is reasonable to expect progress for the case of partial GW*-algebras, since then the powerful theory of von Neumann algebras is available. In particular, the modular theory of Tomita-Takesaki extends, with suitable modifications, to partial GW*-algebras.

What about physical applications? So far spin systems with long range correlations are essentially the only systems where partial *-algebras have had a impact, although they may be used also in Wightman's approach to (axiomatic) quantum field theory (we refer the reader to the recent monograph [12] for a detailed presentation of these applications). However, the mathematical tool is there and may be developed for its own sake. Future research will decide which physical systems, if any, are complex enough to *require* the use of this approach.

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